Combining Generalization Algorithms in Regular Collapse-Free Theories

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Abstract

We are looking at the generalization problem modulo some equational theories. This problem is dual to the unification problem: given two input terms, we want to find a common term whose respective two instances are equivalent to the original terms modulo the theory.

There exist algorithms for finding generalizations over various equational theories. Our focus is on modular construction of equational generalization algorithms for the union of signature-disjoint theories. Specifically, we consider the class of regular and collapse-free theories, showing how to combine existing generalization algorithms to produce specific solutions in these cases.

Additionally, we identify a class of theories that admit a generalization algorithm based on the application of axioms to resolve the problem. To define this class, we rely on the notion of syntactic theories, a concept originally introduced to develop unification procedures similar to the one known for syntactic unification. We demonstrate that syntactic theories are also helpful in developing generalization procedures that are similar to those used for syntactic generalization.

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1 Introduction

The problem of generalization of two terms s and t asks to find a term r such that s and t are substitution instances of r. The generalizations of interest are those that maximally retain similarities between the given terms while abstracting their differences in a uniform way. In other words, they should be the least general ones among generalizations of the given terms. The generalization problem is also called the problem of anti-unification since it can be seen as a dual to unification: while unification focuses on making two expressions more specific by finding their common instance via unifiers, anti-unification abstracts them to a more general form.

The first generalization algorithms were developed in the 1970s [37, 40], motivated by the usefulness of this technique for inductive reasoning. Nowadays, application areas of anti-unification are quite diverse, including, e.g., automated program repair [8, 19, 43],

software code or specification synthesis [21, 39, 48], code similarity detection and change analysis [47, 9, 32], learning-related tasks [18, 35, 38], natural language processing [3, 24, 44], indexing/compression [12, 36, 25], just to name a few. Generalization computation techniques have been investigated for various structures (see, e.g., the recent survey [16] for an overview), among them, for first-order equational theories, which is also the subject of this paper.

In equational anti-unification, syntactic equality is replaced by equality modulo the given equational theory. As a consequence, the given terms do not necessarily have a single least general generalization. Instead, problems are characterized by their minimal complete set of generalizations, which might be a singleton, a finite, an infinite set, or might not exist at all due to a clash between the minimality and completeness requirements. Several authors considered (first-order) equational anti-unification, studying the problem in associative, commutative, associative-commutative, unital, idempotent, absorptive theories, semirings, theories that lead to regular congruence classes, etc [1, 2, 4, 5, 11, 13, 14, 15, 30, 46].

A more general question is related to anti-unification in a combined equational theory: Is it possible to derive a generalization algorithm for a union of equational theories from the existing generalization algorithms for the component theories? This is called the *combination problem for generalization algorithms* and is very important for applications. To the best of our knowledge, our work is the first attempt to address the modular construction of generalization algorithms.

On the other hand, the combination problem has been studied quite intensively for unification; see, e.g., [7, 20, 22, 26, 27, 29, 41, 42, 45]. These modularity results impose certain restrictions on the component theories to guarantee the existence and good properties of the combined algorithm, the main restriction being that the theories are signature-disjoint. Considering regular collapse-free theories is also a classical restriction. In the class of regular collapse-free theories, the left-hand side and the right-hand side of any equational axiom are non-variable terms with exactly the same variables. A combination method is known for unification algorithms in any union of disjoint regular collapse-free theories [45], and another one exists for matching algorithms in any union of disjoint regular theories [34] where regular theories may have collapse axioms between a non-variable left-hand side and a variable right-hand side. It is important to notice that when we relax the restriction on equational theories, we may need to impose a stronger restriction on the kind of algorithms available in the component theories. Hence, the combination method introduced in [7] for the unification problem works in any union of disjoint theories, but as a counterpart, it assumes the existence of general unification procedures capable of dealing with free function symbols.

Our contributions

We consider the combination problem for generalization algorithms in the union of equational theories E_1 and E_2 over the signatures Σ_1 and Σ_2 and a set of free constants C, such that:

- $\Sigma_1 \cap \Sigma_2 = \emptyset$, i.e., the theories are signature-disjoint (and $E_i \cap C = \emptyset$ holds for $i = 1, 2$, since C consists of free constants);
- both E_1 and E_2 are regular collapse-free;
- for each E_i, $i = 1, 2$, there exists a complete generalization algorithm $\mathfrak{G}_i$ which
  - handles ground terms built over $\Sigma_i \cup C$, and
  - for any given ground terms t and u returns a triple (r, ϕ, ψ), where r is a generalization of (t, u) with respective *solved* substitutions (ϕ, ψ), meaning that application of (ϕ, ψ) to r does not lead to further (more specific) generalizations.

For such equational theories E_1 and E_2, we design a combination algorithm for generalization problems in $E_1 \cup E_2$, which relies on E_i-generalization algorithms $\mathfrak{G}_i$ for $i = 1, 2$, and show its completeness. This is our main result, which we further refine in two ways, as explained below.

1. The class of finite theories satisfy the above-mentioned conditions and we show that combined generalization problems in the class can be solved in a modular way. Furthermore, we provide a rule-based generalization algorithm for the subclass of finite syntactic theories. The notion of syntactic theories was originally introduced to develop unification procedures similar to the one known for syntactic unification. We demonstrate that syntactic theories are also helpful in developing a rule-based generalization algorithm that extends syntactic generalization. This extension is obtained by introducing a new mutation rule where the resolvent axioms of the theory are applied to simplify the generalization problems.
2. For theories that admit rule-based generalization algorithms working on configurations of a special form, we design a combination algorithm that uses the rules of the component algorithms (instead of using them as black-boxes), thus obtaining a combined rule-based algorithm with rules of the same form as the component algorithms. We call it the white-box combination method. This leads to a modular result: a combined algorithm obtained in such a way can be used as a component algorithm for further white-box combinations. This white-box combination method is applicable to any instance of the rule-based generalization algorithm introduced for finite syntactic theories.

Organization

After this introduction, Section 2 presents the required background. Then, Section 3 presents some results connecting equational generalization to free generalization, showing that equational generalization in finite theories is reducible to free generalization, and so in finite theories equational generalization is finitary. In Section 4, we discuss the combination of generalization for regular collapse-free theories. Then, Section 5 explores the class of finite syntactic theories by showing a rule-based generalization algorithm. Section 6 considers the problem of combining rule-based generalization algorithms. Finally, Section 7 concludes and briefly discusses possible future work.

2 Preliminaries

2.1 Terms and Substitutions

We consider a first-order alphabet consisting of a signature Σ (set of fixed arity function symbols) and a set of variables $\mathcal{V}$. The set of terms $\mathcal{T}(\Sigma, \mathcal{V})$ over Σ and $\mathcal{V}$ is defined in the standard way: $t \in \mathcal{T}(\Sigma, \mathcal{V})$ iff t is defined by the grammar $t := x \mid f(t_1, \ldots, t_n)$, where $x \in \mathcal{V}$ and $f \in \Sigma$ is an n-ary symbol with $n \geq 0$.

We denote arbitrary function symbols by f, g, h, constants by a, b, c, variables by x, y, z, v, and terms by s, t, r, u. The notions of *term depth*, *term size*, and a *position in a term* are defined in the standard way, see, e.g., [6]. By $t|_p$, we denote the subterm of t at position p, and by $t[s]_p$, a term obtained from t by replacing the subterm at position p with the term s. For any position p in a term t (including the root position ϵ), $t(p)$ is the symbol at position p. The set of all variables in t is denoted by $Var(t)$. A term is called *linear* if no variable occurs in it more than once, and *ground* if $Var(t) = \emptyset$.

A sequence of terms $t_1, \ldots, t_n$ may be written $\bar{t}$ when n is clear from the context.

Given any signature Σ and any finite set of constants C such that $\Sigma \cap C = \emptyset$, the signature $\Sigma \cup C$ is denoted by Σ^C.

A *substitution* is a mapping from $\mathcal{V}$ to $\mathcal{T}(\Sigma, \mathcal{V})$, which is the identity almost everywhere. We use the Greek letters $\sigma, \vartheta, \varphi$ to denote substitutions, except for the identity substitution, which is written as Id. We represent substitutions using the usual set notation, e.g., $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ where $\{x_1, \ldots, x_n\}$ is the domain of σ, denoted by $Dom(\sigma)$. *Application* of a substitution σ to a term t, denoted by $t\sigma$, is defined as $x\sigma = \sigma(x)$ and $f(t_1, \ldots, t_n)\sigma = f(t_1\sigma, \ldots, t_n\sigma)$. Substitution *composition* is defined as a composition of mappings. It is associative but not commutative, with Id playing the role of the unit element. We write $\sigma\vartheta$ for the composition of σ with ϑ.

2.2 Equational Theories

Let E be a set of term pairs, i.e., elements of $\mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$. An *equational Σ-theory generated by E*, denoted by $=_E$, is the least congruence relation on $\mathcal{T}(\Sigma, \mathcal{V})$ that is closed under substitution application and contains E. We call E *a presentation* of $=_E$, and the elements of E are called the *axioms* of $=_E$, written as $l = r$. If $s =_E t$, we say that s is equal modulo E to t. When $E = \emptyset$, every term is equal only to itself, and the theory is called the *empty* or *free* theory.

It is common to slightly abuse the terminology in the literature by calling both E and $=_E$ an equational theory. For a given E, we denote by $\mathsf{sig}(E)$ the set of all function symbols occurring in E.

A sequence of E-equalities $s_1 =_E t_1, \ldots, s_n =_E t_n$ may be written $\bar{s} =_E \bar{t}$ when n is clear from the context.

An axiom $l = r$ is *regular* if $Var(l) = Var(r)$. An axiom $l = r$ is *collapse-free* if l and r are non-variable terms. An axiom $l = r$ is *shallow* if variables can only occur at a position at depth at most 1 in both l and r. An equational theory is *regular* (resp., *collapse-free*/*root-preserving*/*shallow*) if all its axioms are regular (resp., collapse-free/root-preserving/shallow). An equational theory E is *finite* if, for each term t, there are only finitely many terms s such that $t =_E s$. An equational theory E is *subterm collapse-free* if there are no terms t, s such that $t =_E s$ and s is a strict subterm of t. A subterm collapse-free theory is necessarily collapse-free, and a finite theory is necessarily regular and subterm collapse-free.

A theory E is *syntactic* if it has a finite *resolvent presentation* RP, defined as a finite set of axioms RP such that each equality $t =_E s$ has an equational proof $t \leftrightarrow^*_{RP} s$ with at most one equational step $\leftrightarrow_{RP}$ applied at the root position. Any shallow theory is syntactic [17], and any collapse-free theory with finitary unification is syntactic [28].

One can easily check that $A = \{x*(y*z) = (x*y)*z\}$ (Associativity), $C = \{x*y = y*x\}$ (Commutativity), and $AC = A \cup C$ (Associativity-Commutativity) are regular and collapse-free. Moreover, A, C and AC are finite and syntactic [33, 28], but only C is shallow. Since A, C, and AC are finite theories, they are also subterm collapse-free.

2.3 Combination of Equational Theories

In the rest of the paper, we assume that E_i is a regular collapse-free Σ_i-theory for $i = 1, 2$, where Σ_1 and Σ_2 are disjoint signatures. For sake of brevity, the combined signature $\Sigma_1 \cup \Sigma_2$ is abbreviated to $\Sigma_{1,2}$, and the combined theory $E_1 \cup E_2$ is also denoted by $E_{1,2}$.

A Σ-rooted term t is a term whose root symbol is in Σ. Given any $\Sigma^C_{1,2}$-term t where t is Σ_i-rooted, an *alien* subterm of t is any subterm s of t which is not Σ_i-rooted and such that any proper superterm of s in t is Σ_i-rooted. The set of alien subterms of t is

denoted by $Alien(t)$. Given any term t, its *height of theory* is formally defined as follows: $ht(t) = 1 + \max\{ht(t') \mid t' \in Alien(t)\}$. Given a finite set T of ground $\Sigma^C_{1,2}$-terms, we consider $Aliens = \bigcup_{t\in T} Alien(t)$ and a mapping $\pi : Aliens \to FC$ such that FC is a set of fresh free constants ($FC \cap C = \emptyset$) and for any $a, a' \in Aliens$, $\pi(a) = \pi(a')$ iff $a =_{E_1\cup E_2} a'$. Conversely, we define $\pi^{-1} : FC \to Aliens$ as a mapping such that for any $a \in Aliens$, $\pi^{-1}(\pi(a)) =_{E_1\cup E_2} a$. The i-abstraction of any term $t \in T$ is denoted by t^{π_i} and is inductively defined as follows:

- $c^{\pi_i} = c$ if $c \in C$,
- $(f(t_1, \ldots, t_n))^{\pi_i} = f(t_1^{\pi_i}, \ldots, t_n^{\pi_i})$ if $f \in \Sigma_i$,
- $(f(t_1, \ldots, t_n))^{\pi_i} = \pi(f(t_1, \ldots, t_n))$ if $f \in \Sigma_j$ for $j \neq i$.

▶ **Lemma 1** (Correspondence for Equality). *Let E_1 and E_2 be two signature-disjoint regular collapse-free theories. For any Σ^C_i-rooted terms s and t, $s =_{E_1\cup E_2} t$ iff $s^{\pi_i} =_{E_i} t^{\pi_i}$.*

Lemma 1 holds when the component theories E_1 and E_2 are regular collapse-free. In the general case where the component theories are arbitrary, we need to assume that s and t are in layer-reduced form in order to have the same correspondence with their i-abstractions [23].

2.4 Generalization Problems

Given an equational theory E, a term s is *more general than t modulo E*, or is an *E-generalization of t*, if there exists a substitution σ such that $s\sigma =_E t$. In such a case, we write $s \preceq_E t$ and also say the term t is *less general than s modulo E* or that t is an *E-instance* of s. The relation $\preceq_E$ is a quasi-order and generates the equivalence relation, denoted by $\simeq_E$. The strict part of $\preceq_E$ is denoted by $\prec_E$.

An equational generalization problem is formulated as follows:

Given: an equational theory E and two terms s and t;
Find: a term r such that $r \preceq_E s$ and $r \preceq_E t$ (r is said to be an *E-generalization* of s and t).

Given an equational theory E and two terms s and t, a set of terms $\mathcal{G}$ is called a *complete set of E-generalizations* of s and t if it satisfies the following properties:

1. Soundness: every element of $\mathcal{G}$ is an E-generalization of s and t,
2. Completeness: for every E-generalization q of s and t there exists $r \in \mathcal{G}$ such that $q \preceq_E r$.

The set $\mathcal{G}$ is called a *minimal complete set of E-generalizations* of s and t, denoted $mcsg_E(s, t)$ if it, in addition, satisfies the following:

3. Minimality: if there exist $r, q \in \mathcal{G}$ such that $r \preceq_E q$, then $r = q$.

The existence and cardinality of $mcsg_E$ define what is called the *generalization type of an equational theory E* as follows: E has the generalization type 0 (or is *nullary*) if there are two terms s and t such that $mcsg_E(s, t)$ does not exist. Otherwise, the generalization type of E is

- unitary, if $mcsg_E(s, t)$ is singleton for all s and t (in this case, the sole element of $mcsg_E(s, t)$ is called a least general E-generalization of s and t and is denoted by $lgg_E(s, t)$),
- finitary, if $mcsg_E(s, t)$ is finite for all s and t, and there exists at least one pair of terms for which this set is not singleton,
- infinitary, if there exist s and t for which $mcsg_E(s, t)$ is infinite.

In the rest of the paper, we consider generalization algorithms defined as terminating inference systems transforming anti-unification triples (AUTs, for short) written $x : t \triangleq u$ where x is a variable and t, u are the two ground terms to generalize. More precisely, we are focusing on rule-based systems working on configurations $\mathcal{C}$ of the form $A; S; \sigma$, where A and S are sets of AUTs, called the *active set* and *store* of $\mathcal{C}$, respectively, and σ is a substitution, called the *generalization component* of $\mathcal{C}$. From now on, $\mathcal{C}$ is said to be a *PSS-configuration*; it consists of a problem together with a store and a substitution. Given a set of AUTs $P = \bigcup_{i=1}^{n} \{x_i : t_i \triangleq u_i\}$, we associate two substitutions: $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ is the *left* substitution of P and $\{x_1 \mapsto u_1, \ldots, x_n \mapsto u_n\}$ is the *right* substitution of P. A set of AUTs may be written using the disjoint union symbol $\uplus$ to isolate a specific AUT from that set.

3 Equational Generalization vs Free Generalization

The following two theorems characterize equational generalizations with the help of free least general generalizations.

▶ **Theorem 2** (Free *lgg*'s versus E-Generalizations). *Given an equational theory E and two terms s and t, if a term r is an E-generalization of s and t, then there exist terms s', t', and r' such that $s' =_E s$, $t' =_E t$, r' is a free least general generalization of s' and t', and $r \preceq_\emptyset r'$.*

Proof. From r being an E-generalization of s and t we get the existence of two substitutions σ and ϑ such that $r\sigma =_E s$ and $r\vartheta =_E t$. Take $s' = r\sigma$ and $t' = r\vartheta$. Then we have $s' =_E s$ and $t' =_E t$. Moreover, r is a free generalization of s' and t'. Then there exists a free least general generalization r' of s' and t', and we get $r \preceq_\emptyset r'$. ◀

▶ **Theorem 3** (Completeness and Finiteness through Free *lgg*'s). *Let E be an equational theory, s and t be two terms, and $\mathcal{G}$ be the set of terms $\mathcal{G}_E(s,t) = \{lgg_\emptyset(s', t') \mid s' \in [s]_E, t' \in [t]_E\}$. Then*

1. *$\mathcal{G}_E(s,t)$ is a complete set of E-generalizations of s and t.*
2. *$\mathcal{G}_E(s,t)$ is not necessarily minimal.*
3. *If E is a finite theory, then $\mathcal{G}_E(s,t)$ is finite modulo variable renaming. Consequently, every finite theory E has the E-generalization type at most finitary.*
4. *For any E, if $[s]_E$ is finite, then $\mathcal{G}_E(s,t)$ (as well as $\mathcal{G}_E(t,s)$) is also finite modulo variable renaming.*

Proof.

1. Every element $r' \in \mathcal{G}$ is an E-generalization of s and t, because there exist substitutions σ and ϑ such that $r'\sigma = s' =_E s$ and $r'\vartheta = t' =_E t$. Thus, together with Theorem 2, we may conclude that for every E-generalization r of s and t there exists an E-generalization r' of s and t such that $r' \in \mathcal{G}$ and $r \preceq_\emptyset r'$, which implies the completeness of $\mathcal{G}$.
2. Let E be the equational theory that asserts the commutativity of f, and take $s = f(a,b)$ and $t = f(a,b)$. Then $\mathcal{G}_E(s,t) = \{f(a,b), f(x,y)\}$, because $f(a,b) = lgg_\emptyset(f(a,b), f(a,b))$ and $f(x,y) = lgg_\emptyset(f(a,b), f(b,a))$. Obviously, $f(x,y)$ is more general than $f(a,b)$ and, hence, $\mathcal{G}_E(s,t)$ is not minimal.
3. If E is a finite theory, then $[s]_E$ and $[t]_E$ are finite sets. Free least general generalizations are unique (modulo variable renaming). Hence, $\mathcal{G}_E(s,t)$ is finite modulo renaming. Since E-matching is finitary in any finite theory E, $\mathcal{G}_E(s,t)$ can always be minimized, and we get that E-generalization type in finite theories is at most finitary.

4. First, note that the depth of a free generalization of two terms never exceeds the minimum of their depths, and such a generalization does not contain any function symbol that does not appear in both of them. Therefore, for a fixed $s' \in [s]_E$, the depth of $lgg_\emptyset(s', t')$ is bounded by the depth of s', and the set of function symbols that potentially may appear in $lgg_\emptyset(s', t')$ is restricted to the set of function symbols of s'. Hence, there can be only finitely many terms (over a fixed-arity alphabet like ours) that satisfy these conditions. Hence, whatever t' we take, the candidates for $lgg(s', t')$ may come only from a fixed finite (modulo variable renaming) set that depends on s' only: the elements of this set are terms whose free instance is s'. Therefore, for the given s', the set of all terms $\{lgg_\emptyset(s', t') \mid t' \in \mathcal{T}(\mathcal{F}, \mathcal{V})\}$ is finite. Since by assumption $[s]_E$ is finite, we get that $\mathcal{G}_E(s, t)$ is a finite union of finite sets, which implies that it is finite.

◀

▶ **Example 4.** This example illustrates an infinite $\mathcal{G}_E$ for E that is non-finite subterm-collapse free (the simplest class of non-finite theories according to [10]). Let $E = \{f(a, g(x)) = f(a, x), f(b, g(x)) = f(b, x)\}$, $s = f(a, c)$, and $t = f(b, d)$. Then

$$\mathcal{G}_E(s, t) = \{f(x, y), f(x, g(y)), f(x, g(g(y)), \ldots\}.$$

In fact, this set contains an infinite chain

$$f(x, y) \prec_E f(x, g(y)) \prec_E f(x, g(g(y)) \prec_E \cdots .$$

4 Combined Generalization in Regular Collapse-Free Theories

In the section, we consider regular collapse-free theories E. With the assumption that E is regular, the following property holds: given any generalization r of a pair of ground terms (t, u), any substitutions ϕ, ψ such that $r\phi =_E t$ and $r\psi =_E u$ are necessary ground. This assumption is useful in the next definition, where the considered substitutions are necessarily ground substitutions.

▶ **Definition 5** (Generalizations with Solved Instantiations). *Let E be a regular Σ-theory. Given any pair of ground Σ^C-terms (t, u), an* E-generalization of (t, u) with respective instantiations (ϕ, ψ) *is a Σ^C-term r together with a pair of substitutions (ϕ, ψ) such that $Var(r) = Dom(\phi) = Dom(\psi)$, $r\phi =_E t$ and $r\psi =_E u$. Given any pair of ground Σ^C-terms (t, u), a triple representing an* E-generalization of (t, u) with solved instantiations (an E-GSI triple of (t, u), for short) *is any triple (r, ϕ, ψ) such that*

- *r is an E-generalization of (t, u) with respective instantiations (ϕ, ψ),*
- *for any $x \in Var(r)$, there is no non-variable E-generalization of $(x\phi, x\psi)$,*
- *for any $x, y \in Var(r)$, $x\phi =_E y\phi$ and $x\psi =_E y\psi$ implies $x = y$*[1].

A set ST of E-GSI triples of (t, u) is said to be complete *if $\bigcup_{(r,\phi,\psi)\in ST}\{r\}$ is a complete set of E-generalization of (t, u). We say that E admits* an algorithm computing a complete set of E-generalizations with solved instantiations (GSI algorithm, for short) *if there exists a computable function returning, for each input pair of ground Σ^C-terms (t, u), a complete set of E-GSI triples of (t, u) denoted by $GSI_E(t, u)$.*

[1] This third condition may be dropped when considering *linear* generalizations.

A GSI algorithm can be constructed via the repeated application of a generalization algorithm together with a matching algorithm. In the case of a finite theory, the generalization problem is finitary and the matching problem as well. Consequently, this iteration can be implemented, and it is necessarily terminating in the case of a finite theory since the size of any generalization is necessarily bounded. Thus, we have that:

▶ **Lemma 6.** *Any finite theory admits a GSI algorithm.*

In Section 5, we introduce another class of theories admitting a GSI algorithm. This class includes non-finite theories.

In general, for the theories we want to combine, we could imagine deriving GSI algorithms from existing rule-based generalization algorithms transforming PSS-configurations (see Section 6). Then, the resulting GSI algorithms can be combined directly as black-boxes, provided that the theories are regular collapse-free.

▶ **Theorem 7** (Modular Property for GSI Algorithm)**.** *The class of regular collapse-free theories admitting a GSI algorithm and with decidable equality is closed by disjoint union.*

Proof. The rule-based procedure given in Figure 1 can be shown terminating thanks to a complexity measure defined as follows: to each configuration $(A; S; \sigma)$, we associate a pair $(mht(A), |S|)$ where $mht(A)$ is the multiset of heights of theory $\bigcup_{(x':t' \triangleq u') \in A} \{ht(t'), ht(u')\}$, and $|S|$ is the cardinality of S. These pairs are ordered lexicographically, the first component being ordered by the multiset extension of the (Noetherian) ordering on positive integers, while the second component is ordered by the (Noetherian) ordering on positive integers. One can easily verify that E_i**-Gen** and $E_{1,2}$**-Sol** strictly decrease the first component of the pair, and $E_{1,2}$**-Mer** does not increase the first component but strictly decreases the second one.

The normal forms with respect to the rule-based procedure are the configurations of the form $(\emptyset; S; \sigma)$ upon which $E_{1,2}$**-Mer** does not apply. Indeed, any configuration $(A; S; \sigma)$ with $A \neq \emptyset$ is necessarily reducible by either E_i**-Gen** or $E_{1,2}$**-Sol**.

For any derivation starting with the input configuration $(\{x : t \triangleq u\}, \emptyset, Id)$ and leading to a final configuration $(\emptyset; S; \sigma)$, we can associate a tuple $(x\sigma, \phi_S, \psi_S)$ where

- $\phi_S = \{y \mapsto t' \mid y : t' \triangleq u' \in S, y \in Var(x\sigma)\}$,
- $\psi_S = \{y \mapsto u' \mid y : t' \triangleq u' \in S, y \in Var(x\sigma)\}$.

Then, the set of all such tuples defines a $GSI_{E_1 \cup E_2}(t, u)$ as stated in Definition 5. This is a consequence of the following facts:

- E_i**-Gen** is sound and complete by Lemma 8 and Corollary 9 (see below),
- $E_{1,2}$**-Sol** is sound and complete since E_1 and E_2 are collapse-free.

◀

We now show the completeness of the algorithm presented in Figure 1 by considering any generalization of the given terms over the combined theory $E_1 \cup E_2$ and showing that we can construct from it an E_i-generalization, where i depends on the root symbol, that uses the same variable instantiations as the combined generalization. In essence, our combination algorithm builds on the completeness of the algorithms for the individual theories and is thus complete as well.

The following technical lemma and corollary are used in the proof of Theorem 7.

E_i-Gen: Generalization in the E_i-theory

$$\{x : t \triangleq u\} \uplus A; S; \sigma \Longrightarrow (P \setminus S_i) \cup A; S_i \cup S; \sigma\{x \mapsto (r\pi^{-1})\},$$

where

- $t(\epsilon) \in \Sigma_i^C, u(\epsilon) \in \Sigma_i^C$,
- $(r, \phi, \psi) \in GSI_{E_i}(t^{\pi_i}, u^{\pi_i})$,
- $P = \{y : (y\phi)\pi^{-1} \triangleq (y\psi)\pi^{-1} \mid y \in Var(r)\}$,
- $S_i = \{y : t' \triangleq u' \mid (y : t' \triangleq u') \in P, t'(\epsilon) \in \Sigma_i^C, u'(\epsilon) \in \Sigma_i^C\}$.

$E_{1,2}$-Sol: Solving in the combined theory

$$\{x : t \triangleq u\} \uplus A; S; \sigma \Longrightarrow A; S \cup \{x : t \triangleq u\}; \sigma, \qquad \text{if } t(\epsilon) \in \Sigma_i \text{ and } u(\epsilon) \in \Sigma_{3-i}.$$

$E_{1,2}$-Mer: Merging in the combined theory

$$\emptyset; \{x : t \triangleq u, x' : t' \triangleq u'\} \uplus S; \sigma \Longrightarrow \emptyset; S \cup \{x : t \triangleq u\}; \sigma\{x' \mapsto x\},$$

if $t =_{E_1 \cup E_2} t'$ and $u =_{E_1 \cup E_2} u'$.

Figure 1 Combined Generalization Procedure for Regular Collapse-Free Theories

▶ **Lemma 8** (Correspondence for Generalization). *Let E_1 and E_2 be two signature-disjoint regular collapse-free theories over respectively the signatures Σ_1 and Σ_2, t and u any Σ_i^C-rooted terms for some $i = 1, 2$, and r any $E_1 \cup E_2$-generalization r of (t, u) with respective instantiations (ϕ, ψ). Consider a bijective mapping $\Pi_V : Alien(r) \to V$ where V is a set of fresh variables and the following terms and substitutions:*

- *r_i is the i-pure term obtained from r by replacing each $r' \in Alien(r)$ by $\Pi_V(r')$.*
- *ϕ_i and ψ_i are two substitutions such that $Dom(\phi_i) = Dom(\psi_i) = Var(r_i)$ and defined as follows:*
  - *for any $x \in Var(r_i) \setminus Var(r)$, $x\phi_i = ((\Pi_V^{-1}(x))\phi)^{\pi_i}$ and $x\psi_i = ((\Pi_V^{-1}(x))\psi)^{\pi_i}$,*
  - *for any $x \in Var(r)$, $x\phi_i = (x\phi)^{\pi_i}$.*

Then, r_i is an E_i-generalization of (t^{π_i}, u^{π_i}) with respective instantiations (ϕ_i, ψ_i), and for any $x \in Var(r_i) \setminus Var(r)$, $\Pi_V^{-1}(x)$ is an $E_1 \cup E_2$-generalization of $(x\phi_i)\pi^{-1}, (x\psi_i)\pi^{-1})$ with respective instantiations (ϕ, ψ).

Proof. By assumption, $r\phi =_{E_1 \cup E_2} t$ and $r\psi =_{E_1 \cup E_2} u$. By Lemma 1, $(r\phi)^{\pi_i} =_{E_1 \cup E_2} t^{\pi_i}$ and $(r\psi)^{\pi_i} =_{E_1 \cup E_2} u^{\pi_i}$. Then, by definition of r_i, ϕ_i and ψ_i, we have $r_i\phi_i = (r\phi)^{\pi_i}$ and $r_i\psi_i = (r\psi)^{\pi_i}$. Consequently, $r_i\phi_i =_{E_1 \cup E_2} t^{\pi_i}$ and $r_i\psi_i =_{E_1 \cup E_2} u^{\pi_i}$.

For each $x \in Var(r_i) \setminus Var(r)$, by definition of ϕ_i and ψ_i, we have that $(\Pi_V^{-1}(x)\phi)^{\pi_i} = x\phi_i$ and $(\Pi_V^{-1}(x)\psi)^{\pi_i} = x\psi_i$. Applying π^{-1} on these identities, we get $(\Pi_V^{-1}(x)\phi)^{\pi_i}\pi^{-1} = (x\phi_i)\pi^{-1}$ and $(\Pi_V^{-1}(x)\psi)^{\pi_i}\pi^{-1} = (x\psi_i)\pi^{-1}$. By definition of π^{-1}, $s =_{E_1 \cup E_2} s^{\pi_i}\pi^{-1}$ for any term s. Consequently, $(\Pi_V^{-1}(x))\phi =_{E_1 \cup E_2} (x\phi_i)\pi^{-1}$ and $(\Pi_V^{-1}(x))\psi =_{E_1 \cup E_2} (x\psi_i)\pi^{-1}$. ◀

▶ **Corollary 9.** *Let E_1 and E_2 be two signature-disjoint regular collapse-free theories over respectively the signatures Σ_1 and Σ_2, and any Σ_i^C-rooted terms t, u where $i = 1, 2$. Then, there is no non-variable $E_1 \cup E_2$-generalization of (t, u) iff there is no non-variable E_i-generalization of (t^{π_i}, u^{π_i}).*

5 Rule-Based Generalization in Finite Syntactic Theories

In this section, we explore a class of equational theories admitting a rule-based generalization algorithm similar to the one known for the empty theory. For the class of finite syntactic theories, we construct a rule-based algorithm fully parametrized by the axioms of a resolvent presentation of a finite syntactic theory.

▶ **Example 10.** The class of finite syntactic theories includes A, C, AC, finite shallow theories [17] such as $f(x) = g(x)$, and permutative theories closed by paramodulation [31]. Notice that $\{f(a) = a\}$ and $\{f(f(x)) = f(x)\}$ are syntactic but not finite theories.

▶ **Lemma 11.** *For any finite syntactic theory E with a resolvent presentation RP, the inference system given in Figure 4 provides an E-generalization algorithm, using the axioms of RP to*

- *check E-equality (Figure 2),*
- *solve E-matching problems (Figure 3),*
- *control the construction of E-generalizations.*

Proof. The rule-based procedure given in Figure 4 terminates since E is assumed to be finite. Indeed, the size of the generalizations associated with configurations is necessarily bounded, and so there are only finitely many applications of **Mut**. The rule **Sol** is sound and complete: when an AUT cannot be transformed by **Mut** or **Dec**, we cannot have a generalization rooted by a function symbol, a variable is the only possible way to get a generalization for this AUT. The soundness and completeness of **Mut** and **Dec** follow from the fact that RP is a resolvent presentation of E.

For any input PSS-configuration $(\{x : t \triangleq u\}; \emptyset; Id)$ where t, u are ground Σ^C-terms, the procedure terminates by computing finitely many PSS-configurations of the form $(\emptyset; S; \sigma)$ and the union of all the terms $x\sigma$ corresponds to a complete set of E-generalizations of $t \triangleq u$, since all the rules are sound and complete. ◀

▶ Remark 12. For the inference system given in Figure 2, an input problem $\{s = t\}$ is such that $s =_E t$ iff all the normal forms reached by this inference system are the empty set.

For the inference system given in Figure 3, the set of solutions of an input E-matching problem $\{s \leq t\}$ consists of all the normal forms reached by this inference system corresponding to solved forms $\bigcup_{i=1}^{m} \{x_i \leq t_i\}$ where $x_1, \ldots, x_m$ are pairwise distinct variables.

Mut$_=$:

$$\{f(\bar{t}) = g(\bar{u})\} \uplus \Gamma \Longrightarrow \{\bar{t} = \bar{l}\theta\} \cup \Gamma$$

where $f(\bar{l}) = g(\bar{r})$ is a fresh renaming of an axiom in RP, and θ is a substitution such that $\bar{r}\theta =_{RP} \bar{u}$.

Dec$_=$:

$$\{f(\bar{t}) = f(\bar{u})\} \uplus \Gamma \Longrightarrow \{\bar{t} = \bar{u}\} \cup \Gamma \qquad \text{where } f \in \Sigma^C.$$

Figure 2 Procedure for Checking Equality in a finite theory with a resolvent presentation RP

The class of finite syntactic theories is known to be closed by disjoint union [33]. Actually, for a union of two disjoint finite theories with respective resolvent presentations RP_1 and RP_2, we have that $RP_1 \cup RP_2$ is a resolvent presentation and it corresponds to a finite theory.

Mut$_{\leq}$: Mutation rule for matching

$\{f(\bar{t}) \leq^? g(\bar{u})\} \uplus \Gamma \Longrightarrow \{\bar{t} \leq^? \bar{l}\theta\} \cup \Gamma$

where $f(\bar{l}) = g(\bar{r})$ is a fresh renaming of an axiom in RP, and θ is a substitution such that $\bar{r}\theta =_{RP} \bar{u}$.

Dec$_{\leq}$: Decomposition rule for matching

$\{f(\bar{t}) \leq^? f(\bar{u})\} \uplus \Gamma \Longrightarrow \{\bar{t} \leq^? \bar{u}\} \cup \Gamma$ where $f \in \Sigma^C$.

Mer$_{\leq}$: Merging rule for matching

$\{x \leq^? t, x' \leq^? t'\} \uplus \Gamma \Longrightarrow \{x \leq^? t\} \cup \Gamma$ if $t =_{RP} t'$.

Figure 3 Matching Procedure in a finite theory with a resolvent presentation RP

Mut: Mutation rule for generalization

$\{x : f(\bar{t}) \triangleq g(\bar{u})\} \uplus A; S; \sigma \Longrightarrow \{y : y\theta \triangleq y\theta' \mid y \in \bar{y}\} \cup A; S; \sigma\{x \mapsto h(\bar{y})\}$

where $h(\bar{l}) = f(\bar{r}) \in RP$, $h(\bar{l}') = g(\bar{r}') \in RP$, $h(\bar{y})\theta =_{RP} f(\bar{t})$ and $h(\bar{y})\theta' =_{RP} g(\bar{u})$.

Dec: Decomposition rule for generalization

$\{x : f(\bar{t}) \triangleq f(\bar{u})\} \uplus A; S; \sigma \Longrightarrow \{\bar{y} : \bar{t} = \bar{u}\} \cup A; S; \sigma\{x \mapsto f(\bar{y})\}$

where $f \in \Sigma^C$ and $\bar{y}$ are fresh variables.

Sol: Solving rule for generalization

$\{x : t \triangleq u\} \uplus A; S; \sigma \Longrightarrow A; S \cup \{x : t \triangleq u\}; \sigma$ if neither **Mut** nor **Dec** applies.

Mer: Merging rule for generalization

$\emptyset; \{x : t \triangleq u, x' : t' \triangleq u'\} \uplus S; \sigma \Longrightarrow \emptyset; S \cup \{x : t \triangleq u\}; \sigma\{x' \mapsto x\}$

if $t =_{RP} t'$ and $u =_{RP} u'$.

Figure 4 Generalization Procedure in a finite theory with a resolvent presentation RP

Consequently, the inference system given in Figure 4 can be directly applied to any union of disjoint finite syntactic theories for which the resolvent presentations are known. However, finding a resolvent presentation may be a difficult task, especially if E-unification is not known to be finitary. In the case we already have a GSI algorithm for a finite theory E, we can combine it directly using the approach introduced in Section 4; we don't have to find a resolvent presentation (if there exists one for E).

▶ Remark 13. The empty theory $\emptyset$ is a finite syntactic theory and its resolvent presentation is empty. In the case $RP = \emptyset$, one can notice that the inference system from Figure 4 corresponds to the classical rule-based generalization algorithm known for the empty theory.

6 White-Box Combination of Rule-Based Generalization Algorithms

In this section, we focus on the modular construction of generalization algorithms transforming PSS-configurations. Let us now define precisely this kind of rule-based generalization algorithm.

▶ **Definition 14.** *Let E be a regular theory. An* E-generalization algorithm transforming PSS-configurations *is an inference system $\mathfrak{G}$ such that for any initial configuration $(\{x : t \triangleq u\}, \emptyset, Id)$, it derives a finite set FC of final configurations $(\emptyset; S; \sigma)$ for which*

$$\bigcup_{(\emptyset;S;\sigma)\in FC} \{x\sigma\}$$

is a complete set of E-generalizations of (t, u). Furthermore, for any configuration $(A; S; \sigma)$ derived from the initial configuration $(\{x : t \triangleq u\}; \emptyset; Id)$, the following (invariant) properties must hold:

- *$x\sigma$ is an E-generalization of (t, u) with respective instantiations (ϕ, ψ) where ϕ (resp., ψ) is the left (resp., right) substitution of $A \cup S$,*
- *for any $x' : t' \triangleq u'$ in S, there is no non-variable E-generalization of (t', u').*

▶ Remark 15. An E-generalization algorithm transforming PSS-configurations leads to a GSI-algorithm. Consider an initial configuration $(\{x : t \triangleq u\}, \emptyset, Id)$ and any triple $(r\sigma, \phi_S, \psi_S)$ such that $(\emptyset, S, \sigma) \in FC$ where FC is the set of final configurations given in Definition 14 and ϕ_S (resp., ψ_S) denotes the left (resp., right) substitution of S. Then, $\bigcup_{(\emptyset,S,\sigma)\in FC}\{(x\sigma, \phi_S, \psi_S)\}$ is actually a $GSI_E(t, u)$.

The rule-based procedure given in Figure 4 corresponds to a generalization algorithm transforming PSS-configurations, thus leading to a GSI algorithm. Conversely, note that a GSI algorithm can be turned into a generalization algorithm transforming PSS-configurations where all derivations are of length 1.

▶ **Corollary 16.** *A theory admits a generalization algorithm transforming PSS-configurations iff it admits a GSI algorithm.*

Generalization algorithms transforming PSS-configurations correspond to GSI algorithms, and so they can be combined using the method given in Figure 1, provided that the respective theories E_1 and E_2 are regular collapse-free. Since this method is given as an inference system also manipulating PSS-configurations, it can be reworded as an $E_1 \cup E_2$-generalization algorithm transforming PSS-configurations. In this new presentation, we do not directly use the triples from GSI_{E_i}-sets, but we just apply one step of the inference system corresponding to an E_i-generalization algorithm transforming PSS-configurations, see Figure 5. This is sufficient to move forward with a solution.

Given two generalization algorithms $\mathfrak{G}_1$ and $\mathfrak{G}_2$ transforming PSS-configurations, let $\mathfrak{G}_1 \oplus \mathfrak{G}_2$ be the inference system given by the set of rules {E_i**-Step**, $E_{1,2}$**-Sol**, $E_{1,2}$**-Mer**} where E_i**-Step** is introduced in Figure 5 while $E_{1,2}$**-Sol** and $E_{1,2}$**-Mer** are from Figure 1.

E_i**-Step:**

$$\{x : t \triangleq u\} \uplus A; S; \sigma \Longrightarrow (A_i\pi^{-1}) \cup A; (S_i\pi^{-1}) \cup S; \sigma(\sigma_i\pi^{-1})$$

where $t(\epsilon) \in \Sigma_i^C, u(\epsilon) \in \Sigma_i^C$ and $(\{x : t^{\pi_i} \triangleq u^{\pi_i}\}; \emptyset; Id) \longrightarrow_{\mathfrak{G}_i} (A_i; S_i; \sigma_i)$.

Figure 5 Rule calling a generalization algorithm $\mathfrak{G}_i$ transforming PSS-configurations.

▶ **Theorem 17** (Modular Construction for Rule-Based Generalization)**.** *Let E_1 and E_2 be two signature-disjoint regular collapse-free theories. If $\mathfrak{G}_i$ is an E_i-generalization algorithm transforming PSS-configurations for $i = 1, 2$, then $\mathfrak{G}_1 \oplus \mathfrak{G}_2$ is an $E_1 \cup E_2$-generalization algorithm transforming PSS-configurations.*

Proof. The inference system $\mathfrak{G}_1 \oplus \mathfrak{G}_2$ is terminating since $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are terminating, and there are finitely many alternations of calls to $\mathfrak{G}_1$ and to $\mathfrak{G}_2$ (see Lemma 8). Actually, $\mathfrak{G}_1 \oplus \mathfrak{G}_2$ corresponds to an $E_1 \cup E_2$-generalization algorithm transforming PSS-configurations since the E_i-**Step** rule from Figure 5 is sound and complete by Lemma 8 and Corollary 9. ◀

▶ Remark 18. In both rules E_i-**Step** and E_i-**Gen**, the constant abstraction $(\cdot)^{\pi_i}$ and the term concretization $(\cdot)\pi^{-1}$ should be considered as transparent operations. With $(\cdot)^{\pi_i}$, aliens are viewed as free constants, with the restriction that two $E_1 \cup E_2$-equal aliens must be viewed as the same free constant. Conversely, with $(\cdot)\pi^{-1}$, these free constants are viewed back as terms in the combined theory $E_1 \cup E_2$. By a slight abuse of notation, if we omit to apply these transformations, then the rules E_i-**Step** and E_i-**Gen** become easy to express.

Note that for free, associative, and commutative theories, all four algorithms for the combined theories $(\emptyset)(A)$, $(\emptyset)(C)$, $(A)(C)$, $(\emptyset)(A)(C)$ described in [2] can be obtained as a combination of generalization algorithms transforming PSS-configurations.

▶ **Example 19.** Consider $E_1 = \{f(x) = g(x)\}$ and $E_2 = C(+)$. For $i = 1, 2$, one can show the existence of an E_i-generalization algorithm transforming PSS-configurations, say $\mathfrak{G}_i$, given by the set of rules

$$\{E_i\text{-}\mathbf{Mut},\ \mathbf{Dec},\ \mathbf{Sol},\ \mathbf{Mer}\},$$

where **Dec**, **Sol**, **Mer** can be found in Figure 4, and the mutation rules E_1-**Mut** and E_2-**Mut** are defined as follows:

E_1-**Mut:**
$$\{x : f(t) \triangleq g(u)\} \uplus A; S; \sigma \Longrightarrow \{y : t \triangleq u\} \cup A; S; \sigma\{x \mapsto f(y)\}$$

E_2-**Mut:**
$$\{x : t + t' \triangleq u + u'\} \uplus A; S; \sigma \Longrightarrow \{y : t \triangleq u', z : t' \triangleq u\} \cup A; S; \sigma\{x \mapsto y + z\}$$

Let us now detail how the combined generalization algorithm $\mathfrak{G}_1 \oplus \mathfrak{G}_2$ is applied to solve

$$x : f(f(a) + f(b)) \triangleq g(f(c) + f(a)).$$

The derived PSS-configurations are listed in the table below, where the rightmost column shows the configuration number, while the first column indicates how the configuration in that row is obtained (which generalization algorithm is applied to which configuration):

	$(\{x : f(f(a) + f(b)) \triangleq g(f(c) + f(a))\}; \emptyset; Id)$	(0)
$\mathfrak{G}_1(0)$	$(\{y_1 : f(a) + f(b) \triangleq f(c) + f(a)\}; \emptyset; \sigma_0\{x \mapsto f(y_1)\})$	(1)
$\mathfrak{G}_2(1)$	$(\{y_2 : f(a) \triangleq f(c), y_3 : f(b) \triangleq f(a)\}; \emptyset; \sigma_1\{y_1 \mapsto y_2 + y_3)\})$	(2)
$\mathfrak{G}_1(2)$	$(\{y_4 : a \triangleq c, y_3 : f(b) \triangleq f(a)\}; \emptyset; \sigma_2\{y_2 \mapsto f(y_4))\})$	(3)
$\mathfrak{G}_1(3)$	$(\{y_3 : f(b) \triangleq f(a)\}; \{y_4 : a \triangleq c\}; \sigma_3)$	(4)
$\mathfrak{G}_1(4)$	$(\{y_5 : b \triangleq a\}; \{y_4 : a \triangleq c\}; \sigma_4\{y_3 \mapsto f(y_5)\})$	(5)
$\mathfrak{G}_1(5)$	$(\emptyset; \{y_4 : a \triangleq c, y_5 : b \triangleq a\}; \sigma_5)$	(6)
$\mathfrak{G}_2(1)$	$(\{y_6 : f(a) \triangleq f(a), y_7 : f(b) \triangleq f(c)\}; \emptyset; \sigma_1\{y_1 \mapsto y_6 + y_7)\})$	(7)
$\mathfrak{G}_1(7)$	$(\{y_8 : a \triangleq a, y_7 : f(b) \triangleq f(c)\}; \emptyset; \sigma_7\{y_6 \mapsto f(y_8)\})$	(8)
$\mathfrak{G}_1(8)$	$(\{y_7 : f(b) \triangleq f(c)\}; \emptyset; \sigma_8\{y_8 \mapsto a\})$	(9)
$\mathfrak{G}_1(9)$	$(\{y_9 : b \triangleq c\}; \emptyset; \sigma_9\{y_7 \mapsto f(y_9)\})$	(10)
$\mathfrak{G}_1(10)$	$(\emptyset; \{y_9 : b \triangleq c\}; \sigma_{10})$	(11)

Hence, (6) and (11) are the final configurations:

- $(\emptyset; \{y_4 : a \triangleq c, y_5 : b \triangleq a\}; \{x \mapsto f(y_1), y_1 \mapsto y_2 + y_3, y_2 \mapsto f(y_4), y_3 \mapsto f(y_5)\})$,
- $(\emptyset; \{y_9 : b \triangleq c\}; \{x \mapsto f(y_1), y_1 \mapsto y_6 + y_7, y_6 \mapsto f(y_8), y_8 \mapsto a, y_7 \mapsto f(y_9)\})$.

The resulting generalizations are:

- $f(f(y_4) + f(y_5))$, with respective instantiations $(\{y_4 \mapsto a, y_5 \mapsto b\}, \{y_4 \mapsto c, y_5 \mapsto a\})$,
- $f(f(a) + f(y_9))$, with respective instantiations $(\{y_9 \mapsto b\}, \{y_9 \mapsto c\})$.

Since both $\mathfrak{G}_1$ and $\mathfrak{G}_2$ can handle any AUT between two constants in C, $\mathfrak{G}_2$ could be applied instead of $\mathfrak{G}_1$ on (3), (5), (8) and (10).

7 Conclusion

We have introduced a combination method for the generalization problem in any union of disjoint regular collapse-free theories. Regularity is required in our framework in order to satisfy that any matching problem has only ground solutions. With this property, we have only to consider the generalization of ground terms, involving possibly free constants. Collapse-freeness is also very useful to solve in an easy way the generalization of two terms rooted in distinct theories. In future work, we plan to relax this regular and collapse-free assumption. Considering (particular) non-disjoint unions of theories would be also a natural continuation, but this seems to be challenging even if only constants are shared. Until now, we have considered rule-based generalization for finite theories, and the next step is to apply our rule-based approach to non-finite theories. Finally, we are interested in investigating how our combination approach could be adapted to generalization procedures that are not necessarily terminating.

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