**Noname manuscript No.**
(will be inserted by the editor)

# Combinatorial Applications of the Compactness Theorem

**Fabián Fernando Serrano Suárez**[1] · **Mauricio Ayala-Rincón**[2] · **Thaynara Arielly de Lima**[3]

Received: date / Accepted: date

**Abstract** This work discusses applications of a formalization in Isabelle/HOL of the Compactness theorem for propositional logic. The formalization of the Compactness theorem is based on the model existence theorem approach and is explained in detail. The applications cover extensions of combinatorial theorems over countable structures, including the De Bruijn-Erdös Graph coloring theorem for countable graphs, König's lemma, and set- and graph-theoretical versions of Hall's theorem for countable families of sets and graphs. The main distinguishing feature of the formalization of these applications is the proof methodology based on the construction of models to apply the Compactness theorem.

## 1 Introduction

The propositional Compactness theorem is of principal importance for any meta-logical development because of the myriad of applications in logic and combinatorics. Typically, this theorem is presented as a simple consequence of the completeness theorem. However, constructive proofs based on Henkin's-style model existence theorem provide the mathematical machinery to design proofs of combinatorial properties in areas such as set theory and graph theory through the construction of logical interpretations and models.

This paper discusses a formalization in Isabelle/HOL of the Compactness theorem for propositional logic according to Smullyan's approach given in the third chapter of his influential textbook on mathematical logic [55], and based on Henkin's model existence theorem. The formalization follows the impeccable presentation in Fitting's

Second and third authors supported by FAPEG 202310267000223 and CNPq Universal 409003/2021-2 grants. The second author is partially funded by CAPES PrInt 001 and CNPq grant 313290/2021-0.

✉ Mauricio Ayala-Rincón E-mail: ayala@unb.br.

[1]Universidad Nacional de Colombia - Sede Manizales E-mail: ffserranos@unal.edu.co · [2]Universidade de Brasília E-mail: ayala@unb.br · [3]Universidade Federal de Goiás E-mail: thaynaradelima@ufg.br

textbook [15] and was initially developed as part of Serrano's thesis [47]. Two other formalizations of the Compactness theorem by Berghofer [5] and by Michaelis and Nipkow [37] were developed in Isabelle. The current formalization uses Berghofer's technique to enumerate (predicate) formulas, fulfilling formula enumeration requirements constructively according to Fitting's textbook [15]. The formalization by Michaelis and Nipkow also includes a proof of the Compactness theorem. Still, it focuses on various properties of different propositional proof systems such as natural deduction, sequent calculus, resolution, and Hilbert systems (see additional discussion in the related work's Section 4). The distinguished feature of the current formalization is related to the formalization of applications of the Compactness theorem outside the logical setting. Specifically, this paper discusses how models and interpretations are built for proving landmark combinatorial results such as König's lemma and the de Bruijn-Erdös $k$-coloring theorem and Hall's theorem, both for the countable infinite case. The technique applied in the formalizations of each of these three properties consists of the construction of a set of propositional formulas specifying the target property (e.g., set-representability, graph-coloration, tree paths); afterward, building models for the countable cases; then, applying the Compactness theorem for concluding the property. Other approaches for addressing König's lemma are available in the Isabelle libraries. For instance, Lochbihler [34] and Traytel and Popescu presented treatments for this result applying coinductive techniques [6] (see the related work's Section 4).

The proofs described in this paper add to the meta-logic available in Isabelle/HOL, another proof of the Compactness theorem for propositional logic for the countable case. The formalizations of the de Bruijn-Erdös $k$-coloring theorem for countable graphs and of König's lemma by the model construction technique and application of the Compactness theorem are unpublished and discussed in detail. The formalization of Hall's theorem for countable sets is only briefly addressed since it was reported in detail in [48]. Also, a graph-theoretical version of Hall's theorem for countable graphs was presented in [49].

### 1.1 Organization

Initially, Section 2 discusses the formalization of the Compactness theorem; afterward, Section 3 details the three applications mentioned above; finally, after a brief discussion on related work in Section 4, Section 5 concludes. When pertinent, the paper includes links to crucial aspects of the development available as [50] through the link https://www.isa-afp.org/entries/Prop_Compactness.html.

## 2 Compactness Theorem

For the preamble of this section, we present a selection of comments extracted from the excellent discussion on the Compactness theorem by Paseau and Leek [39] and on Gödel's mathematical work in [14].

### 2.1 Proofs of the Compactness Theorem

The Compactness theorem is a fundamental property for the model theory of (classical) propositional and first-order logic. Besides algebra and combinatorics, the Compactness theorem also has implications in topology and foundations of mathematics. In general, it implies that any compact logic extending first-order logic cannot express the notions of finitude or infinitude (of a model). Also, it implies that any first-order theory of arithmetic satisfied by the standard model has a non-standard model. It also can be used to prove the Order-Extension Principle: any partial order may be extended to a linear order.

Also, according to Paseau and Leek [39], the first proof of a Compactness theorem for countable versions of propositional and first-order logic was published by Gödel: (Satz X in [22]), who also proved the general version for arbitrary languages applying transfinite recursion in [23]. Mal'cev [36] also proved the Compactness theorem for propositional logic, again using the full strength of the Axiom of Choice. His proof relies on transfinite induction.

The first explicit, published proofs of a Compactness theorem from completeness, which is the one presented in several contemporary textbooks in logic, were given independently by Henkin and Robinson for the first-order functional calculus [28] and [43], and for the simple theory of types [29]. Indeed, Paseau and Leek [39] adequately remark that "proofs of compactness via completeness are not satisfactory because they are based on properties incidental to the semantic property of interest. Such proofs conclude compactness, a semantic property, from a property of the logic relating its syntax to its semantics." The authors also cited Keisler's connections between ultraproducts technique and compactness and essential and unessential applications of such method. In particular, Keisler's viewpoint about such proofs of compactness [32]: "Unlike the completeness theorem, the Compactness theorem does not involve the notion of a formal deduction, and so it is desirable to prove it directly without using that notion." They finish with the following commentary: "From the perspective of a model theorist who sees talk of syntax as a heuristic for the study of certain relations between structures that happen to have syntactic correlates, proving compactness via completeness is tantamount to heresy ([40], page 53)."

Our formalization uses Smullyan's approach in Fitting's textbook [15], which is a direct proof of the Compactness theorem for propositional logic without using the Completeness theorem.

## 2.2 Formalization of the Compactness Theorem

The formalization was first given in [47] and follows Smullyan's proof as presented in Fitting's famous textbook [15]. König's lemma can be used to prove the Compactness theorem for propositional logic in the countable case. Consider a set of formulas $S$. It is enough to order the countable set of sentences in $S$, say as $F_1, F_2, \ldots$, and to build a countable tree with successful evaluations of the propositional letters (i.e., atoms) validating the subsets of formulas $\{F_1\}, \{F_1, F_2\}$, and so on. The infinite branch gives an interpretation of $S$. Other proofs of this theorem are also provided as part of a collection of classical propositional formalizations aiming at applications and teaching logic. For instance, Michaelis and Nipkow developed a formalization, part of IsaFOL, based on an enumeration of all formulas and saturation accordingly to Enderton's textbook proof ([13]) [37]. The formalization in this paper is based on the so-called "model existence theorem". It shows first Hintikka's lemma: Hintikka sets are satisfiable. Such sets are *downward saturated* sets of propositional formulas to be discussed in explaining the formalization.

Berghofer developed a formalization in Isabelle/HOL of the model existence theorem for first-order logic according to Fitting's textbook in [5]. The current formalization is in Isabelle/Isar, aiming for a detailed presentation of the proof. In particular, it follows Berghofer's approach to specify and enumerate (propositional) formulas, formula consistency, and Hintikka sets and to formalize Hintikka's lemma.

The satisfiability of any Hintikka set is proved by assuming a valuation that maps all propositional letters in the set to *true* and all other propositional letters to *false* (mapping *IH* in the theory `HintikkaTheory`). The second step consists of proving that families of sets of propositional formulas, which hold the so-called "propositional consistency property[1]." This property is specified as the property *consistenceP* in the theory `Closedness`. That families of sets satisfying the consistency property consist of satisfiable sets is indeed the model existence theorem (theory `ModelExistence`).

The Model Existence theorem allows using the abstract concept of propositional consistency property to formalize meta theorems in Propositional Logic. This notion abstracts the characteristics associated with the concept of consistency relative to a specific proof procedure, which is used to demonstrate the system's completeness.

The model existence theorem compiles the essence of completeness: a family of sets of propositional formulas that holds the propositional consistency property can be extended, preserving this property to a set collection that is closed for subsets and satisfies the *finite character property*. The finite character property states that a set belongs to the family if and only if each of its finite subsets belongs to the family. With the model existence theorem in hands, the Compactness theorem (theory `PropCompactness`) is obtained easily: given a set of propositional formulas $S$ such that all its finite subsets are satisfiable, one considers the family $\mathcal{C}$ of subsets in $S$ such that all their finite subsets are satisfiable. $S$ belongs to the family $\mathcal{C}$ and the latter holds the propositional consistence property.

**Theorem 1 (Model Existence (Theorem 3.6.2 in [15]))** *If $\mathcal{C}$ is a propositional consistency property, and $S \in \mathcal{C}$, then $S$ is satisfiable.*

**Theorem 2 (Compactness Theorem (Theorem 3.6.3 in [15]))** *Let $S$ be a set of propositional formulas. If every finite subset of $S$ is satisfiable, so is $S$.*

We present the most important definitions and proofs used in the formalization.

The language of propositional formulas is specified through the datatype *formula* in the theory `SyntaxAndSemantics`. This datatype defines formulas according to the standard propositional grammar:

$$formula = \bot \mid \top \mid atom \mid \neg.\, formula \mid formula \,\Box.\, formula$$

where $\Box \in \{\wedge, \vee, \rightarrow\}$. So, formulas are built inductively from the constants $\bot$, $\top$, and a set of atoms, *negation, conjunction, disjunction* and *implication.* Notice that it is necessary to discriminate meta and object logical connectives. For this, the connectives in the grammar are succeeded by a dot: "$\neg.$", "$\wedge.$", "$\vee.$", and "$\rightarrow.$".

In this grammar, it is not assumed as in [15] that the set of atoms is countable infinite. Working without this assumption in our formalization makes it precise when

[1] A family of sets is a *propositional consistency property* if no set includes a propositional letter and its negation, no set includes the constant *false* or the negation of the constant *true*; if the double negation of a formula, $\neg\neg F$, belongs to a set in $S$, then $S \cup \{F\}$ belongs to the family; if a formula $F_1 \wedge F_2 \in S$ then the set $\{F_1, F_2\} \cup S$ belongs to the family; and if the formula $F_1 \vee F_2 \in S$ then $\{F_1\} \cup S$ or $\{F_2\} \cup S$ belongs to the family.

the hypothesis on the enumerability of the set of formulas is essential. In particular, this assumption is required to prove the model existence theorem but neither to prove Hintikka's lemma nor to determine the maximality of Hintikka sets.

Also, in the theory `SyntaxAndSemantics`, to evaluate the *truth-value* of propositional formulas over an interpretation, we specify the operator *t-v-evaluation* as usual, using evaluations for all kinds of formulas in the datatype *v-truth*. Since the evaluation should reflect the mathematical object under study, and the meta-logic to prove theorems over this object-logic is the one of Isabelle, the evaluation is built over two different elements for *false* and *true*.

For a set of formulas $S$, the notion of *model* is specified as an interpretation such that all formulas in $S$ evaluate (through the application of *t-v-evaluation*) to *true* in this interpretation. And, a set of formulas $S$ is *satisfiable* is specified as a set for which a model exists.

The notion of compactness is specified using the Isabelle specification for finite sets and a specification for countable sets.

The specification of finite sets is imported from the HOL theory for finite sets. In particular, we use the characterization *finite* $A$ if and only if there exist $n \in \mathbb{N}$ and a surjective function $f$ from $\{m \in \mathbb{N} \mid m < n\}$ to $A$. Also, countable sets are specified with the predicate *enumeration* stating the existence of a surjective function with domain $\mathbb{N}$.

A *Hintikka set* $H$ can be understood as a syntactically downward saturated set:

- no atom and its negation can belong to $H$;
- neither $\bot$ nor $\neg.\top$ belong to $H$;
- if $\neg.\neg.F$ belongs to $H$ then $F$ belongs to $H$;
- if $F_1 \wedge . F_2$ ($\alpha$ formula) belongs to $H$, then both $F_1$ and $F_2$ belong to $H$ and,
- if $F_1 \vee . F_2$ ($\beta$ formula) belongs to $H$, either $F_1$ or $F_2$ belongs to $H$.

This property is specified as *HintikkaP* in the theory `HintikkaTheory`. *Hintikka's lemma* states that Hintikka sets are satisfiable: *HintikkaP* $H \longrightarrow$ *satisfiable* $H$.

The formalization of Hintikka's lemma is by induction on the structure of the formulas in a Hintikka set $H$ by applying the technical theorem *HintikkaP_ model_ aux* in the above theory. This theorem applies a series of lemmas to evaluate all possible cases of formulas in the set $H$. Indeed, considering the Boolean evaluation *IH* that maps all atoms in $H$ to *true* and all other letters to *false*, the most interesting cases of the inductive proof are those related to implicational formulas in $H$ and the negation of arbitrary formulas in $H$. These cases are not straightforward since the saturation of implicational and negation formulas is not considered in the definition of Hintikka sets.

For an implicational formula in $H$, say $F_1 \to . F_2$, it is necessary to prove that its evaluation by *IH* is *true*; also, whenever $\neg.(F_1 \to . F_2)$ belongs to $H$, it should be proved that the evaluation of the implicational formula is *false*. The proof is obtained by relating such formulas with $\beta$ and $\alpha$ formulas.

The second interesting case is the one related to arbitrary negations. In this case, it is proved that if $\neg.F$ belongs to $H$, its evaluation by *IH* is *true*, and in the case that $\neg.\neg.F$ belongs to $H$, considered in the definition of Hintikka sets, its evaluation by *IH* is also *true*.

As previously mentioned, these theorems require the definition of *propositional consistency property*. Let $\mathcal{C}$ be a collection of sets of propositional formulas. We call $\mathcal{C}$ a *propositional consistency property* if it meets the conditions given in the definition *consistenceP* for each $S \in \mathcal{C}$, as specified below and provided in the theory `Closedness`. In this definition, *FormulaAlpha* and *FormulaBeta* correspond respectively to conjunctive ($\alpha$) and disjunctive ($\beta$) propositional formulas as defined in [15]. The first and second components of $\alpha$ and $\beta$ formulas are selected with the operators *Comp1* and *Comp2*, respectively. The specification of *ConsistenceP* is included here to clarify how the object grammar and the meta-logic are used by discriminating the Isabelle and the object logical connectives, with and without dots, respectively.

**Definition** :: *'b formula set set* $\Rightarrow$ *bool* **where**
$$
\begin{array}{l}
consistenceP\ \mathcal{C} = \\
\quad (\forall S.\ S \in \mathcal{C} \longrightarrow (\forall P.\ \neg\ (atom\ P \in S \wedge (\neg.atom\ P\ ) \in S)) \wedge \\
\quad \bot\ \notin S \wedge (\neg.\top) \notin S \wedge \\
\quad (\forall F.\ (\neg.\neg.F) \in S \longrightarrow S \cup \{F\} \in\ \mathcal{C}) \wedge \\
\quad (\forall F.\ ((FormulaAlpha\ F) \wedge F{\in}S) \longrightarrow (\ S \cup \{Comp1\ F,\ Comp2\ F\}) \in \mathcal{C}) \wedge \\
\quad (\forall F.\ ((FormulaBeta\ F) \wedge F{\in}S) \longrightarrow (\ S \cup \{Comp1\ F\} \in \mathcal{C}) \vee (\ S \cup \{Comp2\ F\} \in \mathcal{C})))
\end{array}
$$

Since the specified grammar does not restrict the set of atoms to be countable, the specification of the model existence theorem, given in the theory `ModelExistence`, adds the assumption of the existence of an enumeration for the set of formulas:

**assumes** *hyp*: $\exists\, g.\ enumeration\ (g{::}\ nat \Rightarrow\ 'b\ formula)$

The formalization of the model existence theorem requires a series of properties.

In the theory `Closedness`, closedness properties of the propositional consistency property are proved. Such properties allow us to conclude that if the collection of a set of formulas $\mathcal{C}$ holds the property, then $(\mathcal{C}^{+})$, which is the closure of $\mathcal{C}$ under subsets, does it too. See theorem *Closed_ ConsistenceP* in this theory.

The finite character property is specified in the theory `FinitenessClosedCharProp`, as given below.

$$finite\text{-}character\ \mathcal{C} = (\forall S.\ S \in \mathcal{C} = (\forall S'.\ finite\ S' \longrightarrow S' \subseteq S \longrightarrow S' \in \mathcal{C}))$$

To formalize the finite character property for subset closed families of sets of propositional formulas that satisfy the propositional consistency property, it is necessary to show a series of properties for extensions of the families of sets. It is proved that a finite character property of families of sets of propositional formulas implies subset closeness (see lemma *finite_ character_ closed* in the above theory).

Finally, the finite subset closure of a collection of sets $\mathcal{C}$ is denoted as $\mathcal{C}^{-}$ as specified in the definition *closure_ cfinite* in the theory `FinitenessClosedCharProp`. Then, the theorem that states that subset closed propositional consistency properties can be extended to satisfy the finite character property is specified as theorem *cfinite-consistenceP* in the theory stating that

*consistenceP* $\mathcal{C}$ and *subset-closed* $\mathcal{C}$ implies *consistenceP* $\mathcal{C}^{-}$,

where *subset-closed* $\mathcal{C}$ means that for all $S$ an element of $\mathcal{C}$, if $S' \subseteq S$ then $S' \in \mathcal{C}$.

The proof is by induction on the structure of propositional formulas based on the analysis of cases for the possible different types of formula in the sets of the collection of sets $\mathcal{C}$ that hold the propositional consistency property, lemmas *cond_ characterP1* to *cond_ characterP5* in the theory `FinitenessClosedCharProp`.

An interesting corollary of the model existence theorem is that each subset of a set of formulas $\mathcal{C}$ that satisfies the propositional consistency property built over a countable set of propositional letters is satisfiable. This corollary requires proving that the set of formulas built over an enumerable set of propositional letters is enumerable. The last result is formalized in the theory `FormulaEnumeration`. This corollary corresponds to the presentation of the model existence theorem in [15] except that all details concerning the guarantee that the set of formulas built over an enumerable set of atoms is indeed enumerable are made explicit in the formalization.

Now, we discuss the formalization of the Compactness theorem. The auxiliary lemma *ConsistenceCompactness* in the theory `PropCompactness` is required to apply the model existence theorem to obtain the Compactness theorem. This lemma states the general fact that the collection of all sets of propositional formulas such that all their subsets are satisfiable is a propositional consistency property:

$$\mathit{ConsistenceP}\ \{W \mid \forall A\,(A \subseteq W \wedge \mathit{finite}\ A \longrightarrow \mathit{satisfiable}\ A)\}$$

With this lemma in hand, since any countable set of formulas that belongs to $\mathcal{C}$ is satisfiable as a consequence of the model existence theorem, one obtains the formalization of the theorem *Compactness_ Theorem* in the same theory. As the model existence theorem, this theorem requires the assumption that the set of formulas is enumerable. Indeed, given a set $S$ of formulas, all whose finite subsets of formulas are satisfiable, it is only necessary to prove it belongs to the above collection of sets.

So, the key technical part of formalizing the Compactness theorem from the model existence theorem is the auxiliary lemma *ConsistenceCompactness*. This lemma is formalized unfolding the definition *consistenceP* through a series of auxiliary lemmas *consistenceP_ Prop1* to *consistenceP_ Prop6* specialized to each case in the definition of *consistenceP*. For instance, the auxiliary lemma *consistenceP_ Prop5* states the required satisfiability property for the case of formulas $\alpha$:

$$\begin{gathered}\forall F.(F \in W \wedge\ \mathit{FormulaAlpha}\ F)\\ \forall A.(A \subseteq W \wedge\ \mathit{finite}\ A) \longrightarrow\ \mathit{satisfiable}\ A\\ \longrightarrow\\ \forall A.(A \subseteq W\{\mathit{Comp1}\ F, \mathit{Comp2}\ F\} \wedge\ \mathit{finite}\ A) \longrightarrow\ \mathit{satisfiable}\ A\end{gathered}$$

This lemma is formalized by applying another auxiliary lemma (such as for the case of the property of formulas $\beta$ in the definition of *consistenceP*), the lemma *satisfiableUnion2* in the theory `PropCompactness` that states the more simple property below.

$$\mathit{FormulaAlfa}\ F \wedge\ \mathit{satisfiable}\ (A \cup \{F\}) \longrightarrow \mathit{satisfiable}\ (A \cup \{\mathit{Comp1}\ F,\ \ \mathit{Comp2}\ F\})$$

Craig's interpolation theorem is another application of the model existence theorem for propositional logic, formalized in Serrano's Thesis ([47]). In addition, and always following Fitting's textbook elegant presentation ([15]), the Thesis includes formalizations in Isabelle/Isar of a variety of results for first-order logic as the model existence theorem and the Löwenheim-Skolem theorem, following the seminal Berghofer's applicative approach. All these results were obtained constructively as applications of the model existence theorem and the completeness of natural deduction.

## 3 Applications of the Compactness Theorem

This section discusses the formalization of three important applications of the Compactness theorem for propositional logic, namely, the de Bruijn-Erdös $k$-coloring theorem, König's lemma, and Hall's theorem.

## 3.1 De Bruijn-Erdös Graph Coloring Theorem

The theory `k_coloring` formalizes the de Bruijn-Erdös $k$-coloring theorem for countable graphs. Since the few required elements from graph theory are basic notions, like definitions of graphs, induced graphs, finite graphs, and coloring, we opt to specify them as part of the theory instead of importing robust available theories on graphs that bring several results that were not crucial for the current formalization, such as, for example, the one developed by Noschinski and Neumann [38].

We start with the definition of digraphs. Digraphs are elements of type *set* and the Cartesian product of the *set* by itself, being the first component of a digraph, the set of vertices $V[G]$, and the second one $E[G]$, the set of edges. So, a graph $G$ satisfies the predicate:

$$is\_\ graph\ G \equiv \forall uv.(u,v) \in E[G] \longrightarrow u \in V[G] \wedge v \in V[G] \wedge u \neq v$$

Notice how the irreflexibility of the edge relation is obtained from the definition above, excluding self-loops.

A pair of vertices $u, v \in V[G]$ is called to be adjacent if $(u,v) \in E[G]$ or $(v,u) \in E[G]$.

The subgraph $H$ induced by a subset of vertices of $G$ is specified as the relation below.

$$\textit{is-induced-subgraph}\ H\, G \equiv V[H] \subseteq V[G] \wedge E[H] = E[G] \cap (V[H] \times V[H])$$

The well-definedness of induced subgraphs is proved as a lemma stating that

$$is\_\ graph\ G \wedge\ \textit{is-induced-subgraph}\ HG \longrightarrow\ is\_\ graph\ H$$

A digraph is $k$-colorable, for $k \in \mathbb{N}$, if its vertices can be mapped to the set $\{1, \ldots, k\}$ avoiding mapping adjacent vertices to the same natural. The concepts of a $k$-coloring and a graph being $k$-colorable are specified as the following predicates.

$$\begin{gathered} coloring\ c\, k\, G \equiv (\forall u.u \in V[G] \longrightarrow c(u) \leq k) \wedge (\forall uv.(u,v) \in E[G] \longrightarrow c(u) \neq c(v)) \\ colorable\ G\, k \equiv \exists c.\ coloring\ c\, k\, G \end{gathered}$$

### *3.1.1 Informal proof of the de Bruijn-Erdös Theorem*

The de Bruijn-Erdös theorem is stated below. The "pen-and-paper" proof applies the Compactness theorem. This version of de Bruijn-Erdös' theorem diverges from the standard one in which the hypothesis refers not only to $k$-coloration of all finite induced subgraphs, but to $k$-coloration of all finite subgraphs, as sketched in [8] (Chapter 18). This difference makes our formalization stronger than the standard proofs that apply compactness.

**Theorem 3 (de Bruijn-Erdös)** *Let $G = (V, E)$ be a countable graph and $k$ be a positive integer. If for all finite $S \subseteq V$, $G_S$ is $k$-colorable, then $G$ is $k$-colorable.*

*Proof* Let us fix a set of propositional symbols,

$$\mathcal{P} = \{C_{u,i} \mid u \in V, 1 \leq i \leq k\}$$

where $C_{u,i}$ is interpreted as "the vertex $u$ has color $i$". We define three propositional formula sets:

1. $\mathcal{F} = \{C_{u,1} \vee C_{u,2} \vee \cdots \vee C_{u,k} \mid u \in V\}$;
2. $\mathcal{G} = \{\neg(C_{u,i} \wedge C_{u,j}) \mid u \in V, 1 \leq i, j \leq k, i \neq j\}$;
3. $\mathcal{H} = \{\neg(C_{u,i} \wedge C_{v,i}) \mid u, v \in V, (u, v) \in E, 1 \leq i \leq k\}$.

The previous sets express the following properties regarding $G$ and $k$, respectively:

1. each vertex corresponds to at least a color;
2. no vertex is associated with more than one color; and,
3. adjacent vertices are associated with different colors.

Let $\mathcal{T} = \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$. The Compactness theorem is applied to prove that $\mathcal{T}$ is satisfiable.

Let $S$ be a finite subset of $\mathcal{T}$ and $V_0 = \{u_1, \ldots, u_n\}$ be the set of all vertices $u$ such that $C_{u,i}$ for some $i$, occurs in some formula in $S$.

Let $G_{V_0} = (V_0, E_0)$ be the subgraph of $G$ induced by $V_0$.

Let $c : V_0 \to [k]$ be a $k$-coloring of $G_{V_0}$.

We define the interpretation $v : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ as

$$v(C_{u,i}) = \begin{cases} \mathsf{T} \text{ if } u \in V_0 \text{ and } c(u) = i, \\ \mathsf{F} \ \ \text{otherwise.} \end{cases}$$

We have $v(F) = \mathsf{T}$ for all $F \in S$ since $c$ is a $k$-coloring and $F \in \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$. Thus, $\mathcal{T}$ is finitely satisfiable; hence, by the Compactness theorem, it is satisfiable.

Let $I : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ be an interpretation that satisfies $\mathcal{T}$. We establish a correspondence $c : V \to [k]$ defined as $c(u) = i$ if and only if $I(C_{u,i}) = \mathsf{T}$.

Therefore, by the definition of $\mathcal{T}$ and since $I(F) = \mathsf{T}$ for all $F \in \mathcal{T}$, one has that $c$ is a $k$-coloring of $G = (V, E)$. Indeed, since $\mathcal{F}$ and $\mathcal{G}$ are satisfiable, to each vertex $v \in V$ corresponds exactly a color in $[k]$, thus, $c$ is a function. Finally, since $\mathcal{H}$ is satisfiable, adjacent vertices have different colors. □

### *3.1.2 Formalization of de Bruijn-Erdös k-Coloring Theorem*

This subsection discusses the details of the formalization in the theory `k_coloring` of the $k$-coloring theorem following the proof sketch given in Theorem 3.

The theory includes the following recursive definition of atomic disjunctions of length $k + 1$ for each vertex $v$. Such disjunctions are required in the construction of the sets of formulas $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and $\mathcal{T}$.

$$\begin{aligned} &\textit{atomic-disjunctions } v\, 0 = \ \textit{atom } (v, 0) \mid \\ &\textit{atomic-disjunctions } v\, (\textit{Suc } k) = (\textit{atom } (v, \textit{Suc } k)) \vee. (\textit{atomic-disjunctions } v\, k) \end{aligned}$$

So, the set of formulas $\mathcal{T}$ is specified as $\mathcal{T} G\ k\ \equiv \mathcal{F} G\ k\ \cup\ \mathcal{G} G\ k\ \cup\ \mathcal{H} G\ k$ , where the formulas $\mathcal{G}$, and $\mathcal{H}$ are specified straightforwardly as below.

$$\mathcal{F} G\ k\ \equiv \bigcup_{v \in V[G]} \textit{atomic-disjunction } v\, k$$

$$\mathcal{G} G\ k\ \equiv \{\neg. (\textit{atom } (v, i) \wedge. \textit{atom } (v, j)) \mid v \in V[G] \wedge 0 \leq i, j \leq k \wedge i \neq j\}$$

$$\mathcal{H} G\ k\ \equiv \{\neg. (\textit{atom } (u, i) \wedge.\ \textit{atom}(v, i)) \mid (u, v) \in E[G] \wedge 0 \leq i, j \leq k\}$$

The set of vertices occurring in a formula is specified recursively on the structure of formulas and used to define the set of vertices occurring in a finite set of formulas as *vertices-set-formulas*. In this manner, the set of vertices in a finite subset of formulas $S \subseteq \mathcal{T}\, G\, k$, denoted as $V_0$ in the proof of Theorem 3, is built.

Several auxiliary lemmas are formalized that relate a subset of formulas $S$, the sets of propositional symbols in $\mathcal{P}$, representing vertices and their possible colors, and the set $\mathcal{T}\, G\, k$.

The subgraph of $G$ induced by a subset of vertices $V \subseteq V[G]$ is specified as

$$\textit{subgraph-aux}\ G\ \ V \equiv (V, E[G] \cap (V \times V))$$

Then, it is possible to formalize that the subgraph induced by the vertices occurring in a finite subset $S$ of formulas in $\mathcal{T}\, G\, k$, denoted as $G_{V_0}$ in Theorem 3, is a finite graph:

Let $S$ be a finite subset of $\mathcal{T}$, and $V_0 = \{u_1, \ldots, u_n\}$ be the set of vertices $u$ such that $C_{u,i}$, for some $i$, occurs in some formula in $S$. From this, it is proved that the subgraph of $G$ induced by $V_0$, $G_{V_0} = (V_0, E_0)$, is also a finite graph. This fact is formalized as lemma *finite-subgraph* below.

$$S \subseteq (\mathcal{T}\, G\, k) \wedge \textit{finite}\, S \longrightarrow \textit{finite-graph}\,(\textit{subgraph-aux}\ G\,(\textit{vertices-set-formulas}\ S))$$

The theorem *coloring-satisfiable*, below, states that a coloring of $G_{V_0}$ enables the construction of a model of $S$.

$$S \subseteq (\mathcal{T}\, G\, k) \wedge \textit{coloring}\, f\ k\,(\textit{subgraph-aux}\ G\,(\textit{vertices-set-formulas}\ S)) \longrightarrow \textit{satisfiable}\ S$$

The formalization of this fact uses the function *graph-interpretation* below. It allows one to show that the function gives a $k$-coloring of the subgraph induced by the vertices in the set of formulas $S$.

$$\textit{graph-interpretation}\, G\, f\ =\ (\lambda(v,i).(\textit{if}\ v \in V[G] \wedge f(v) = i\ \textit{then}\ \mathsf{T}\ \textit{else}\ \mathsf{F}))$$

An interpretation $I : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ that satisfies $\mathcal{T}$ establishes a $k$-coloring $c : V \to [k]$ given by $c(u) = i$ if and only if $I(C_{u,i}) = \mathsf{T}$.

$$\textit{graph-coloring}\ I\ k\ = (\lambda v.(\textit{THE}\ i.(\textit{t-v-evaluation}\ I\,(\textit{atom}\ (v,i)) = \mathsf{T}) \wedge 0 \leq i \leq k))$$

The next step in the formalization is establishing the existence of the *graph-coloring* function when $I$ is a model of $\mathcal{T}$. This fact is formalized using a series of auxiliary lemmas stating the existence and unicity of the color associated with each vertex regarding any interpretation $I$ model of $\mathcal{T}$, summarized in the lemma *coloring-function*:

$$\begin{array}{l} u \in V[G]\ \wedge\ I\ \textit{model}\ (\mathcal{T}\, G\, k) \longrightarrow \\ \exists! i.\ \textit{t-v-evaluation}\ I\,(\textit{atom}\ (u,i) = \mathsf{T}\ \wedge\ 0 \leq i \leq k\ \wedge\ \textit{graph-coloring}\ I\, k\, u = i \end{array}$$

The following main result, theorem *satisfiable-coloring*, establishes that if the set of formulas $\mathcal{T}$ for a graph $G$ and a natural $k$ is satisfied, then $G$ is $k$-colorable:

$$\textit{satisfiable}\ (\mathcal{T}\, G\, k) \longrightarrow\ \ \textit{colorable}\ G\, k$$

The proof assumes a model $I$ for $\mathcal{T}$ by the satisfiability hypothesis. Applying lemma *coloring-function*, the function *graph-coloring* will give a unique color $i$, $0 \leq i \leq k$ for each vertex $u$ in the graph. This happens since the evaluation of the formulas $\mathcal{F}$ and $\mathcal{G}$ for the model $I$ will guarantee the existence of a unique atom $c_{u,i}$ that is *true*. Finally, applying another auxiliary lemma (*distinct-color*), which states that *graph-coloring* gives different colors for adjacent vertices, since $I$ is also a model for $\mathcal{H}$, it is guaranteed that the evaluation of $I$ for adjacent vertices $u$ and $v$ is such that the unique atoms $c_{u,i}$ and $c_{v,j}$ evaluated as *true* are such that $i \neq j$.

To conclude, the de Bruijn-Erdös theorem (Theorem 3), the last theorem formalized in theory `k_coloring`, is proved by applying theorem *coloring-satisfiable* to prove that any finite subgraph $H$ of $G$ induces a finite subset $S$ of formulas of $\mathcal{T}$ that is satisfiable. Therefore, by the Compactness theorem, $\mathcal{T}$ is satisfiable. Consequently, applying the theorem *satisfiable-coloring* $G$ is $k$-colorable.

### 3.2 Formalization of König's Lemma

Using the Compactness theorem for propositional logic, we formalize König's lemma for countable trees:

*Any infinite countable finitely branching tree has an infinite path.*

The formal proof steps, given in the theory `KoenigLemma`, follow the approach sketched in [8].

For this formalization, specialized notions of trees as binary relations are required. Such notions are available in the HOL theory and well-developed and specialized theories in the Archive of Formal Proofs, such as the theory of abstract reduction [56]. But since, for our purposes, only a few related definitions, such as finitely branching tree, level, path on trees, and reachability, are required to build a set of formulas expressing König's lemma, we opt to avoid importing such elaborated theories. Indeed, if we import the abstract reduction theory to specify the notion of a tree as a specialized binary relation, a series of other theories irrelevant to our formalization exercise are imported.

The definitions and properties regarding trees needed to formalize König's lemma are specialized *on* (sub)sets of the domain and range of binary relations:

i) $R$ is *irreflexive on* $A$ iff $\forall x \in A, (x,x) \notin R$.
ii) $R$ is *transitive on* $A$ iff $\forall x,y,z \in A\,((x,y) \in R \wedge (y,z) \in R \longrightarrow (x,z) \in R)$.
iii) $R$ is *total on* $A$ iff $\forall x,y \in A\,(x \neq y \longrightarrow (x,y) \in R \vee (y,x) \in R)$.
iv) An element $a \in A$ is a *minimum* element of $A$ iff $\forall x \in A\,(x \neq a \longrightarrow (a,x) \in R)$.
v) The set of *predecessors* of $a \in A$ is defined as $Pr(a) = \{x \in A \mid (x,a) \in R\}$.

The theory `KoenigLemma` includes the necessary more specialized definitions for the case of interest in which $R$ is a binary relation on $A$ such that for all $a \in A$, the set $Pr(a)$ is finite:

vi) (*height*) For all $a \in A$, the height of $a$, $height(a)$, is the number of its predecessors:

$$height(a) = |Pr(a)|.$$

vii) (*level*) For each integer number $n \geq 0$, the $n$-th level of $R$ is the set of elements of $A$, whose height is $n$; that is,

$$Lv(n) = \{a \in A \mid\ height(a) = n\}.$$

viii) (*imm_succ*) For each $a \in A$, the set of immediate successors of $a$, $imm\_succ(a)$, is defined as

$$imm\_succ(a) = \{y \in A \mid (a,y) \in R \wedge height(y) = height(a) + 1\}.$$

Strict partial and linear orders are defined as usual: let $R$ be a binary relation on $A$. The pair $(A, R)$

1. is a *strict partial order* (SPO) if and only if $R$ is irreflexive and transitive;
2. is a *linear order* if and only if it is an SPO and $R$ is a total relation.

The uniqueness of the minimum in an SPO is given by the following lemma, which is formalized in Isabelle: *let $(A, R)$ be an SPO. If $A$ has a minimum element, then such an element is unique.*

Next, the definition of trees is given.

**Definition 1 (Tree)** Let $R$ be a nonempty binary relation on $A$. The pair $T = (A, R)$ is a tree if and only if

1. $T$ is an SPO;
2. $A$ has a minimum element, which we call the *root* of $T$;
3. For all $a \in A$, the set $Pr(a)$ is finite and the restriction of $R$ to $Pr(a)$ is total.

The elements of $A$ are called the nodes of $T$.

A tree $T = (A, R)$ is *finite* if and only if the set of nodes is finite; otherwise, it is *infinite*. $T$ is *finitely branching* if and only if for each $a \in A$, the set $imm\_succ(a)$ is finite.

**Definition 2 (Path)** Let $T = (A, R)$ be a tree. A set of nodes $B \subseteq A$ is a path of $T$ if and only if $(B, R)$ is a linear order and $B$ is maximal (regarding the subset relation). If $B$ is finite, it is called a finite path; otherwise, $B$ is an infinite path.

Notice that a finitely branching tree having an infinite path has an infinite branch. The specifications of such relations are straightforward. For instance, sub-linear orders and paths in the theory `KoenigLemma` are given below.

$$\textit{sub-linear-order } B\ A\ r \equiv B \subseteq A \wedge (\textit{strict-partial-order } A\, r) \wedge (\textit{total-on } B\ r)$$

$$\textit{path } B\, A\, r \equiv (\textit{sub-linear-order } B\, A\, r) \wedge (\forall C. B \subseteq C \wedge \textit{sub-linear-order } C\, A\, r \longrightarrow B = C)$$

Specifications of *finite-* and *infinite-path* conjugate to *path* $B\,A\,r$ the finiteness and infiniteness of $B$: *finite* $B$, and $\neg$ *finite* $B$.

The following lemmas (Lemmas 1, 2, 3 and 4) are crucial and form the basis to prove König's lemma.

**Lemma 1 (Finiteness of levels in Finitely Branching Trees)** *Let $T = (A, R)$ be a tree. The following statements are equivalent:*

1. *$T$ is finitely branching.*
2. *For all $n \geq 0$, the set $Lv(n)$ is finite.*

Lemma 1 is formalized in the theory `KoenigLemma`, but from this equivalence, only the necessity is essential to formalize König's lemma (namely, the direction 1 implies 2). This fact is formalized as the lemma *finite-level*.

Lemmas 2 and 3 guarantee the existence of a path from any node to the root of a tree and the non-emptiness of each level in a finitely branching infinite tree, respectively. They are formalized as lemmas *path-to-node* and *all-levels-non-empty*.

**Lemma 2 (Root Reachability in Trees)** *Let $T = (A, R)$ be a tree. If $n \geq 0$ and $x \in Lv(n+1)$ then for all $k$, $0 \leq k \leq n$, there is $y_k$ such that $(y_k, x) \in R$ and $y_k \in Lv(k)$.*

**Lemma 3 (Non-emptiness of Levels)** *Consider $T = (A, R)$ a finitely branching infinite tree. Thus, for all $n \geq 0, Lv(n) \neq \emptyset$.*

Finally, Lemma 4, formalized as *emptyness-inter-diff-levels*, states that the elements in the same set of predecessors are at distinct levels.

**Lemma 4 (Emptyness of Level Intersection)** *Let $T = (A, R)$ be a tree. Suppose that $(x, z) \in R$, $(y, z) \in R$, and $x \neq y$. If $x \in Lv(n)$ and $y \in Lv(m)$ then $Lv(n) \cap Lv(m) = \emptyset$.*

*3.2.1 Informal proof of König's Lemma*

In this section, we discuss the "pen-and-paper" proof of König's lemma (Theorem 4) obtained as a consequence of the Compactness theorem and using previous results.

**Theorem 4 (König's Lemma)** *Every finitely branching infinite (countable) tree has an infinite branch.*

*Proof* Let $T = (A, R)$ be a finitely branching infinite countable tree. Consider the following set of propositional symbols indexed by the vertices of $T$:

$$\mathcal{P} = \{B_u \mid u \in A\}.$$

From the set $\mathcal{P}$, one can define a set of formulas $\mathcal{T}$, such that if $\mathcal{T}$ is satisfiable then for any interpretation $I$, which is model of $\mathcal{T}$, the set of vertices $\mathcal{B}$ is an infinite path of $T$:

$$\mathcal{B} = \{u \in A \mid I(B_u) = \mathsf{T}\}$$

$\mathcal{T}$ is given by the union of the following three sets of propositional formulas.

1. For each $n \in \mathbb{N}$,
$$\mathcal{F} = \{ \bigvee_{u \in Lv(n)} B_u \mid n \in \mathbb{N}\},$$
where $\bigvee_{u \in Lv(n)} B_u$ is the disjunction of the atomic formulas corresponding to the elements of the level $Lv(n)$, which is a finite set by the Lemma 1.
2. $\mathcal{G} = \{B_u \longrightarrow B_v \mid u, v \in A, (v, u) \in R\}$,
3. $\mathcal{H} = \{\neg(B_u \wedge B_v) \mid u, v \in Lv(n), u \neq v, n \in \mathbb{N}\}$.

The previous sets allow the characterization of an infinite path in a tree. Indeed, if a set $B$ of vertices of $T$ satisfies such sets, then for any $n \in \mathbb{N}$, there is at least one vertex of $T$ in the level $n$ which belongs to $B$; every predecessor of any element of $B$ belongs to $B$, and $B$ has only a vertex in the level $n$.

Now, we show that the set $\mathcal{T} = \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$ is satisfiable by applying the Compactness theorem.

Let $S$ be a finite subset of $\mathcal{T}$. Since $S$ is finite, the set

$$N = \{u \in A \mid B_u \text{ occurs in some formula of } S\}$$

is also finite; consequently, the set of the heights of vertices from $N$ has a maximum element $h$. Additionally, one has that $Lv(h+1) \neq \emptyset$ since $T$ is infinite and finitely branching (Lemma 3).

Consider $t \in Lv(h+1)$ and define the interpretation $I : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ as,

$$I(B_u) = \begin{cases} \mathsf{T} \text{ , if } (u,t) \in R \\ \mathsf{F} \text{ , otherwise.} \end{cases}$$

Notice that, $I(J) = \mathsf{T}$ for every formula $J \in S$. In fact:

1. If $J \in \mathcal{F}$ then $J = \bigvee_{u \in Lv(n)} B_u$, which corresponds to the disjunction of the atomic formulas associated with the vertices of the level $n$, for some $n \leq h$. Since the vertices that occur in $J$ have height $n < h+1$, there exists $u \in Lv(n)$ such that $(u,t) \in R$ (Lemmas 3, 2). Consequently, $I(B_u) = \mathsf{T}$ and $I(J) = \mathsf{T}$.
2. If $J \in \mathcal{G}$ then there exist $u, w \in A$ such that $J = B_u \longrightarrow B_w$ and $(w,u) \in R$. If $I(J) = \mathsf{F}$ then $I(B_u) = \mathsf{T}$ and $I(B_w) = \mathsf{F}$. Consequently, $(u,t) \in R$ and $(w,t) \notin R$ which is impossible considering that $(w,u) \in R$ and $R$ is transitive relation. Thus, $I(J) = \mathsf{T}$.
3. If $J \in \mathcal{H}$ then there exist $u, w \in Lv(n)$, for some $n \geq 0$, such that $u \neq w$ and $J = \neg(B_u \wedge B_w)$. Since $u$ and $w$ belong to the same level, one has that $(u,t) \notin R$ or $(w,t) \notin R$ (Lemma 4). Consequently, $I(B_u) = \mathsf{F}$ or $I(B_w) = \mathsf{F}$, and $I(J) = \mathsf{T}$.

Therefore, $\mathcal{T}$ is finitely satisfiable and, as a consequence of the Compactness theorem, $\mathcal{T}$ is satisfiable.

Let $I : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ be a model for $\mathcal{T}$. Then,

$$\mathcal{B} = \{u \in A \mid I(B_u) = \mathsf{T}\}$$

is an infinite path of $T$:

Since $I$ satisfies $\mathcal{F}$ and $\mathcal{H}$, one has that, for each level $n$, the intersection $\mathcal{B} \cap Lv(n)$ is a singleton vertex. In the following, we show that $(\mathcal{B}, R)$ is a total and maximal relation and $\mathcal{B}$ is infinite.

(a) $(\mathcal{B}, R)$ is a total relation: consider $u, w \in \mathcal{B}$ such that $u \neq w$. Assume that $\mathit{height}(u) < \mathit{height}(w)$. Let $n = \mathit{height}(u)$ and $x$ be the predecessor of $w$ at level $n$. Then, $B_w \longrightarrow B_x \in \mathcal{G}$, hence $I(B_w \longrightarrow B_x) = \mathsf{T}$. Since $I(B_w) = \mathsf{T}$, $I(B_x) = \mathsf{T}$. Therefore, $x \in \mathcal{B}$ and, since $u, x \in Lv(n)$, one concludes that $u = x$. Thus, $(u,w) \in R$. The case $\mathit{height}(w) < \mathit{height}(u)$ is proved analogously. Therefore, one concludes that $(\mathcal{B}, R)$ is total.

(b) $(\mathcal{B}, R)$ is maximal: we prove that if $\mathcal{B} \subseteq \mathcal{B}'$ and $(\mathcal{B}', R)$ is total then $\mathcal{B}' \subseteq \mathcal{B}$. Let $x \in \mathcal{B}'$, $n = \mathit{height}(x)$ and $u$ be a vertex that belongs to the intersection of $\mathcal{B}$ and the vertices at level $Lv(n)$. Since $u \in \mathcal{B}'$ and $(\mathcal{B}', R)$ is total, if $u \neq x$, then either $(u,x) \in R$ or $(x,u) \in R$, which is impossible since, in a strict order, comparable elements with a finite number of predecessors are at different levels. Therefore, $x = u$, which implies $\mathcal{B}' \subseteq \mathcal{B}$.

(c) $\mathcal{B}$ is infinite: since $I$ satisfies $\mathcal{F}$, it is enough to prove that for all $n \geq 0$, $Lv(n) \neq \emptyset$. This implies that there exists $u$ such that $u \in \mathcal{B} \cap Lv(n)$, therefore, $\mathcal{B}$ is infinite. Suppose there exists $n$ such that $Lv(n) = \emptyset$. This implies that for all $m > n$, $Lv(m) = \emptyset$ too. Consequently, since $T$ is finitely branching, it would be finite. To conclude, one also needs to consider that $Lv(n) \cap Lv(m) = \emptyset$ for all $n \neq m$, and therefore $\bigcup_{n \in \mathbb{N}} \mathcal{B} \cap Lv(n)$ is infinite. □

It is relevant to redundantly stress here that although there are several formalizations of König's lemma, as those discussed in the related work (Section 4.2), the technique of building a set of propositional formulas specifying the existence of infinite paths in trees, then building models for the countable case and finally applying the Compactness theorem has not been formalized before.

### *3.2.2 Formalization of König's Lemma*

In this subsection, we explain the crucial steps in formalizing this proof.

The specification uses the recursive constructor *disjuction-nodes* of the disjunction of atoms below.

$$\begin{aligned}&\textit{disjuction-nodes}\ [\,] = \mathsf{F}\\&\textit{disjuction-nodes}\ (v\#D) = (\textit{atom}\ v) \vee.\,(\textit{disjuction-nodes}\ D)\end{aligned}$$

$\mathcal{T}$ is defined as $\mathcal{T} \equiv (\mathcal{F}\,A\,r) \cup (\mathcal{G}\,A\,r) \cup (\mathcal{H}\,A\,r)$, where the sets of formulas $\mathcal{F}, \mathcal{G}, \mathcal{H}$, and $\mathcal{T}$ are specified below. Notice that $\mathcal{H}$ is built as the union of all the sets $\mathcal{H}n$ of negations of formulas of the form $(B_u \wedge B_v)$ for nodes at the same level $(n)$. The operator *set-to-list* transforms sets into lists.

$$\begin{aligned}&\mathcal{F} \equiv \textstyle\bigcup_n .\{\textit{disjunction-nodes}(\textit{set-to-list}\ (\textit{level}\ A\,r\,n)\}\\&\mathcal{G} \equiv \{(\textit{atom}\ u) \to.\,(\textit{atom}\ u) \mid u, v \in A \wedge (v, u) \in r\}\\&\mathcal{H}_n \equiv \{\neg.\,((\textit{atom}\ u) \wedge.\,(\textit{atom}\ v)) \mid u, v \in (\textit{level}\ A\,r\,n) \wedge u \neq v\}\\&\mathcal{H} \equiv \textstyle\bigcup_n .\mathcal{H}_n\,A\,r\,n\end{aligned}$$

The operator *maximum-height* specifies the maximum height of the nodes occurring in a set of formulas. It uses *nodes-set-formulas*, which defines the union of the nodes in a finite set of formulas.

$$\textit{maximum-height}\ A\,r\,S = \textit{Max}\ (\bigcup x \in \textit{nodes-set-formulas}\ S.\{\textit{height}\ A\,x\,r\})$$

Let $S$ be a set of formulas, and $h$ be the maximum height of the set of nodes occurring in the formulas of $S$. The following function returns a node at level $Lv(h+1)$.

$$\textit{node-sig-level-max}\ A\,r\,S = \textit{SOME}\ u.u \in (\textit{level}\ A\,r\,((\textit{maximum-height}\ A\,r\,S) + 1)$$

The next step in the formalization is proving a lemma, *satisfiable-path*, that specifies that any finite subset $S$ of $\mathcal{T}$ is satisfiable:

$$\textit{infinite-tree}\ A\,r \wedge \textit{finitely-branching}\ A\,r \wedge S \subseteq (\mathcal{T}\,A\,r) \wedge \textit{finite}\ S \longrightarrow \textit{satisfiable}\ S$$

The formalization of the above result consists of building a straightforward model in the following manner: first, one selects a node, say $u$, in the tree at level $h+1$, where $h$ is the maximum level of the set of nodes occurring in the formulas of $S$; then, for the interpretation the truth value of all nodes (atomic formulas) except the predecessors of $u$, which have truth value *true*, is *false*. This is built as the interpretation *path-interpretation* $A\,r = (\lambda v.(\textit{if}\ (v, u) \in r\ \textit{then}\ \mathsf{T}\ \textit{else}\ \mathsf{F}))$.

In this way, using lemmas 3, 2 and 4, in the theory `KoenigLemma`, *all-levels-non-empty*, *path-to-node* and *emptyness-inter-diff-levels*, respectively, one concludes that such an interpretation holds in $S$. Therefore, $\mathcal{T}$ is finitely satisfiable, and so is satisfiable by the Compactness theorem.

The following definition of the set of nodes $\mathcal{B}$, which are *true* in an interpretation $I$, gives the construction of the infinite path used in the proof of König's lemma (Theorem 4): $\mathcal{B}\, A\, I \equiv \{u \mid u \in A \wedge \textit{t-v-evaluation}\ I\,(\textit{atom}\ u) = \mathsf{T}\}$. The following two lemmas describe the crucial properties of $\mathcal{B}$.

The first Lemma, *intersection-branch-set-nodes-at-level*, states that if $\mathcal{B}$ is built from an infinite finitely branching tree and $I$ is an interpretation that satisfies $\mathcal{F}$, then $\mathcal{B}$ has at least a node in each level of the tree. The formalization is obtained by induction on the number of nodes at any tree level.

$$\begin{array}{l}\textit{infinite-tree}\ A\,r \wedge\ \textit{finitely-branching}\ A\,r \wedge \forall F \in (\mathcal{F}\,A\,r).\ \textit{t-v-evaluation}\ I F = \mathsf{T} \\ \longrightarrow \forall n. \exists x. x \in \textit{level}\ A\,r\,n \wedge x \in (\mathcal{B}\,A\,I)\end{array}$$

The second Lemma, *intersection-branch-emptyness-below-height*, states that for any tree and interpretation $I$ that satisfies $\mathcal{H}$, the set $\mathcal{B}$ has at most one node with truth value *true* at each level of the tree. The formalization follows by contradiction.

$$\begin{array}{l}\forall F \in (\mathcal{H}\,A\,r).\ \textit{t-v-evaluation}\ I\,F\ = \mathsf{T} \wedge\ x, y \in (\mathcal{B}\,A\,I) \wedge x \neq y \wedge \\ x \in\ \ \textit{level}\ A\,r\,n \wedge y \in\ \ \textit{level}\ A\,r\,m\ \ \longrightarrow\ \ n \neq m\end{array}$$

From the direct application of the previous two lemmas, one formalizes another result, *intersection-branch-level*, stating that if the tree is an infinite finitely branching tree and the interpretation $I$ is a model of $\mathcal{F}$ and $\mathcal{H}$, the set $\mathcal{B}$ has only one node at each level of the tree:

$$\forall F \in (\mathcal{F}\,A\,r) \cup (\mathcal{H}\,A\,r).\ \textit{t-v-evaluation}\ I\,F\ =\ \mathsf{T} \longrightarrow \forall n. \exists u. (\mathcal{B}\,A\,I) \cap\ \textit{level}\ A\,r\,n = \{u\}$$

The following simple definitional Lemma, *predecessor-in-branch*, states that for any tree and interpretation $I$ that satisfies $\mathcal{G}$, all predecessors of a node in $\mathcal{B}$ also belong to $\mathcal{B}$.

$$\forall F \in (\mathcal{G}\,A\,r).\ \textit{t-v-evaluation}\ I\,F\ =\ \mathsf{T} \wedge y \in (\mathcal{B}\,A\,I) \wedge (x, y) \in r \wedge y \in A \longrightarrow x \in (\mathcal{B}\,A\,I)$$

To conclude, it is necessary to guarantee that for any infinite finitely branching three and an interpretation $I$ of $\mathcal{T}$, $(\mathcal{B}\,A\,I)$ is indeed an infinite path. The first step is proving that it is indeed a path. This fact is specified as lemma *is-path*, formalized by applying all previous lemmas.

$$\begin{array}{l}\textit{infinite-tree}\ A\,r \wedge\ \textit{finitely-branching}\ A\,r \wedge \forall F \in (\mathcal{T}\,A\,r).\ \textit{t-v-evaluation}\ I\,F\ =\ \mathsf{T} \\ \longrightarrow\ \ \textit{path}\ (\mathcal{B}\,A\,I)\,A\,r\end{array}$$

The second step, formalized as lemma *infinite-path*, is to prove the path above, built from a model of $\mathcal{F}$, is indeed infinite.

$$\begin{array}{l}\textit{infinite-tree}\ A\,r \wedge\ \textit{finitely-branching}\ A\,r \wedge \forall F \in (\mathcal{F}\,A\,r).\ \textit{t-v-evaluation}\ I\,F\ =\ \mathsf{T} \\ \longrightarrow\ \ \textit{infinite}\ (\mathcal{B}\,A\,I)\end{array}$$

Finally, the formalization of König's lemma (Theorem 4), the last result in theory `KoenigLemma` ↗, is obtained by firstly applying the lemma *satisfiable-path* that proves that any finite subset $S$ of an infinite finitely branching tree satisfies $\mathcal{T}$. After that, the Compactness theorem is applied to conclude that the tree satisfies $\mathcal{T}$. In the sequence, assuming that $I$ is a model of $\mathcal{T}$ for the tree and building the set $\mathcal{B}$, one concludes that the tree has an infinite path.

## 3.3 Formalizations of Hall's Theorem

This subsection briefly discusses the application of the Compactness theorem in the Isabelle/HOL formalizations of Hall's theorem for countable sets and graphs described in detail in [48] and [49].

The Hall's theorem, also called "marriage theorem," proved primarily by Philip Hall [24], provides necessary and sufficient conditions to choose a distinct representative for each set in a finite family of finite sets $\mathcal{A}$ over elements in a set $S$.

Given $S$, an arbitrary set, and $\{S_i\}_{i\in I}$ a collection of not necessarily distinct subsets of $S$ with indices in the set $I$, a function $f : I \to \bigcup_{i\in I} S_i$ is a *system of distinct representatives* (SDR) for $\{S_i\}_{i\in I}$ if:

1. for all $i \in I$, $f(i) \in S_i$, and;
2. $f$ is an injective function.

From the definition of an SDR, one can state Hall's theorem for sets as follows.

**Theorem 5 (Hall's Theorem | finite case)** *Consider an arbitrary set $S$ and a positive integer $n$. A finite collection $\{S_1, S_2, \ldots, S_n\}$ of finite subsets of $S$ has an SDR if and only if the so-called* marriage condition *(M) below is satisfied.*

*For every $1 \leq k \leq n$ and an arbitrary set of $k$ distinct indices $1 \leq i_1, \ldots, i_k \leq n$, one has that $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$.* *(M)*

Hall's theorem is a landmark result that is equivalent to several other significant theorems in combinatory and graph theory (cf. [7], [8], [42]), namely: Menger's theorem (1929), König's minimax theorem (1931), König–Egerváry theorem (1931), Dilworth's theorem (1950), Max Flow-Min Cut theorem (Ford-Fulkerson algorithm), among others. Consequently, a complete formalization of Hall's theorem gives rise to formally proving those equivalent results. Considering Isabelle/HOL theorem prover, Jiang and Nipkow [30] formalized Hall's theorem by implementing both Halmos and Vaughan's [25] and Rado's [41] techniques.

More general versions of Hall's theorem were established [41]. In particular, Hall's theorem, as enunciated in Theorem 6, holds for a countable collection of finite subsets $\{S_i\}_{i\in I}$ of a set $S$.

**Theorem 6 (Hall's Theorem | countable case)** *Let $S$ be an arbitrary set and $I$ an enumerable set of indices of finite subsets of $S$. The family $\{S_i\}_{i\in I}$ has an SDR if and only if the condition $(M^*)$ below holds.*

*For every finite subset of indices $J \subseteq I$, one has that $|\bigcup_{j\in J} S_j| \geq |J|$.* *$(M^*)$*

This theorem is formalized in the theory `Hall_Theorem`.

As another application of the Compactness theorem for propositional logic, Serrano et al. formalized Theorem 6 in Isabelle/HOL. Such a development combines the formalization of the Compactness theorem as in [47], described in Section 2, and of Jiang and Nipkow's for the finite case of Hall's theorem. The formal proof of the countable case of Hall's theorem in Isabelle/HOL was recently published in [48] and gives rise to provide mechanisms to formally establish general versions of results that are equivalent to Theorem 6.

For instance, besides the set-theoretical version of Hall's theorem for countable families of sets 6, another well-known version, Hall's theorem for graphs, was also formalized.

**Theorem 7 (Hall's Theorem Graph Version | countable case)** *Let $G = \langle X, Y, E\rangle$ be a digraph such that the set of vertices $X \cup Y$ is countable, the set of edges holds $E \subseteq X \times Y$, and for each vertex $x \in X$, the set of neighborhoods of $x$ $N(x) = \{y \mid (x, y) \in E\}$ is finite. Then $G$ contains a perfect matching covering the set of vertices $X$ if and only if ($M^\dagger$) below holds.*

*For every finite subset of vertices $J \subseteq X$, one has that $|\bigcup_{j\in J} N(j)| \geq |J|$.* $(M^\dagger)$

This theorem is formalized in the theory `Hall_Theorem_Graphs`.

Previously, we cited some combinatorial theorems equivalent to Hall's theorem. Depending on the result, the proof of such an equivalence can be adapted to either the set-theoretical or graph-theoretical versions. For example, König–Egerváry theorem states that the minimum cover in a finite bipartite graph has the same cardinality as a maximum matching. Thus, if we assume Hall's theorem for finite graphs, one possible way to infer König–Egerváry theorem will consist of building a reduction from the latter to the former. Considering the nature of König–Egerváry theorem, it is clear that the graph-theoretical version of Hall's theorem is more appropriate than the set version to establish the equivalence between these theorems.

In [49], by applying authors' development in [48], the infinite graph-theoretical version of Hall's theorem was formalized. The mechanization focuses on maintaining specifications and proofs as closely as possible to textbooks since our primary objective was to increase mathematicians' interest in using interactive proof assistants. Although this, the specification also includes a concise and more automatized proof using locales, which can be seen at the end of the theory `Hall_Theorem_Graphs`.

Interestingly, other combinatorial well-known results equivalent to Hall's theorem in the finite case are not straightforwardly equivalent in the infinite case; for instance, the infinite version of König-Egerváry theorem that as reported in [3] cannot be inferred from the Compactness theorem. Thus, another of the aspects we are interested in is to explore if possible restricted variations of infinite versions of König-Egerváry theorem can be obtained as a consequence of the Compactness theorem.

## 4 Related Work

### 4.1 Formalizations of the Compactness Theorem

As mentioned in Subsection 2.2, other proofs in Isabelle/HOL of the Compactness theorem were given by Berghofer [5] and by Michaelis and Nipkow. The former follows Smullyan's presentation [55] as presented in Fitting's textbook [15] in the more general setting of propositional logic. The latter is part of IsaFOL [37]. In general, formalizations of the Compactness theorem belong to collections of developments for propositional and first-order logic, as is the case of IsaFOL (e.g., [17], [46], [16]). In particular, Michaelis and Nipkow formalized proof systems for propositional logic, such as sequent calculus, natural deduction, Hilbert systems, and resolution; they added to IsaFOL proofs of soundness, completeness, cut-elimination, interpolation, and the model existence theorem. However, the formalization of compactness follows a different approach, as the one of this paper, which is based on an enumeration of all formulas and saturation [37]. Michaelis and Nipkow focus on logic properties, and for this, they specify translations between these proof systems, allowing the transferring of soundness and completeness from each of these systems to the others. One of their formalizations of compactness follows Enderton's enumeration and formula saturation approach [13]; from this formalization, they infer completeness for natural deduction and sequent calculus. As in the current paper, building Hintinkka sets and specifying the propositional consistency property, they prove the model existence theorem and formalize compactness as an application of this theorem. Also, directly applying the model existence theorem, they present alternative formalizations of the completeness of the sequent calculus and Hilbert systems.

Among a variety of solid formal developments in classical logic, which provide elements for formalizations of theorems as those treated in this paper, one can include Shankar's pioneering formalizations of the Church-Rosser and the first Gödel incompleteness theorem in the Boyer-Moore theorem prover [51]. Also, it deserves to mention Harrison's formalization in HOL Light of essential results such as the compactness and the Löwenheim-Skolem theorems [26]. Harrison's formalization of the propositional Compactness theorem is also for the countable case and applies Zorn's lemma to extend satisfiable sets to maximal satisfiable sets of propositional formulas (as in the proof given in Enderton's textbook [13]).

### 4.2 Formalizations of König's lemma, and de Bruijn-Erdös and Hall's Theorems

Nowadays, proof assistants include robust proof engines and elaborated mathematical libraries that make the formalization of König's lemma an easy routine exercise. An earlier proof of König's lemma in the Boyer-Moore theorem prover is reported by Kaufmann in [31]. The formalization uses the NQTHM extension of this prover to deal with quantification by applying the technique of (event) Skolemization. The existence of an infinite path in a finitely branching infinite tree is obtained using the predicate "for any node with infinite descendants there exists a successor with infinite descendants." Bancerek developed another earlier formalization of this theorem in Mizar [4]. The formalization states the lemma, proving the existence of an infinite branch whenever the tree has arbitrary long finite chains.

There are no other formalizations in Isabelle/HOL of König's lemma based on compactness. From these formalizations, there are two interesting ones based on coinductive techniques. One of them, by Traytel and Popescu, available in the standard Isabelle distribution (theory `Koenig.thy`), uses definitional commands for codatatypes, corecursion, and conduction implemented in [6], from mechanisms coined out in [57]. It defines "finitely branching countable trees" and proves co-inductively that any infinite tree has infinite paths. The notion of an "infinite" tree is coinductively defined as a tree for which a descendent tree exists that is also "infinite." This allows a straightforward application of coinduction to infer the existence of a König's path (specified as a stream of node labels) for any such "infinite" tree. This approach additionally requires proving that trees with infinite nodes also have König's paths. The second one, by Lochbihler, is given as an example (in the theory `Koenigslemma.thy`) part of the library Coinductive [34]. It first defines infinite finitely-branching connected graphs as connected graphs with an infinite set of nodes and nodes of finite degree. It then coinductively proves that infinite, finitely-branching graphs have infinite paths. Such paths are coinductively defined using conductive lists, also available in this library.

Despite the fact of the existence of excellent libraries on graph theory for different interactive theorem provers (e.g., those related to Gonthier's formalization of the four-color theorem for planar graphs in Coq [18, 20, 19]), to the best of our knowledge there are no formalizations of the de Bruijn-Erdös $k$-coloring theorem, neither for the finite nor for the countable case.

Considering the finite version of Hall's theorem, Romanowicz and Grabowski [44] reported the first formalization of this result in Mizar. Jiang and Nipkow [30] presented two formalizations in Isabelle/HOL: in addition to a formalization of Rado's proof ([41]), also used in Mizar, the Isabelle/HOL development formalizes Vaughan's proof ([25]). Also, a formalization in Coq applies Dilworth's decomposition theorem and bi-partitions in graphs [52]. Dilworth's theorem is formalized in Mizar in [45]. Recently, Gusakov, Mehta, and Miller [21] reported different formalizations in Lean of the finite version of Hall's theorem; the first, in terms of indexed families of finite subsets, the second, in terms of the existence of injections that saturate binary relations over finite sets and, the third, in terms of matchings in bipartite graphs. Related combinatorial results are reported in recent works by Doczkal et al. in their graph theory Coq library (e.g., [10], [12], and [11]). Additionally, Singh and Natarajan formalized in Coq other combinatorial results as the perfect graph theorem and a weak version of this theorem (e.g., [53], [54]).

Adaptations to the infinite case from theorems equivalent to the finite case of Hall's marriage theorem may be elaborated. Moreover, such adaptations would not necessarily be derivable from the Compactness theorem. An example is König's duality theorem that states that in every bipartite graph $G = \langle X, Y, E \rangle$, there exists a matching $M \subseteq E$ such that selecting one vertex from each arc in $M$ one has a *cover* of the graph [1, 3]. This theorem is a strong form of the König-Egerváry theorem, stating that in a finite bipartite graph, the size of a maximal matching is equal to that of a minimal *cover* [33]. The key difference of the duality theorem is that such a cover of the graph cannot be extracted from any matching; namely, given any matching of the graph, it is possible to build a cover of the same cardinality as the cardinality of the matching, but not that covers the graph entirely. So, the notion of *König cover* came to arise, which is defined as a cover of the graph that consists of a selection of one vertex from each arc of a matching.

Lifting results from the finite to the infinite through the application of compactness (of König's lemma) corresponds to a recursive construction of a procedure that produces the target solution in the degree of unsolvability of the halting problem [3]. Such a recursive construction is possible for Dilworth's theorem (restricting the maximal anti-chains in infinite partial ordered sets to be finite - [9], see also Sec. 2.5 in [27]) but not for König's duality theorem. Indeed, Aharoni et al. [3] proved that the complexity of constructing covers exceeds the complexity of the halting problem; it is even a problem of higher complexity than answering all first-order questions about arithmetic. Also, they proved that the Compactness theorem and König's lemma do not suffice to prove the duality theorem and other related results in matching theory.

A remarkable Isabelle recent development by Lochbihler et al. [35] is a formalization of the min-cut max-flow theorem over countable infinite networks following Aharoni et al. proof technique in [2]. The proof technique is not based on applying the Compactness theorem, making exploring alternatives to address this problem and related problems interesting through the approach used in the current paper.

There are two formalizations of the countable set-theoretical version of Hall's theorem: one by the authors detailed in [48], and another by Gusakov, Mehta, and Miller presented in [21]. Also, we formalized a countable graph-theoretical version derived from the set-theoretical formalization presented in [49]. The distinguishing feature of our formalization in Isabelle/HOL is the application of the Compactness theorem. In

**Table 1** Theories of the development - quantitative data

| **Theory Name** | **Line Numbers** | **Number of Proved Formulas** Lemmas | Corollaries | Theorems |
|---|---|---|---|---|
| `SyntaxAndSemantics` | 691 | 17 | | 3 |
| `UniformNotation` | 694 | 29 | | |
| `Closedness` | 180 | 7 | | 1 |
| `FinitenessClosedCharProp` | 337 | 7 | | 2 |
| `MaximalSet` | 235 | 5 | 1 | 4 |
| `HintikkaTheory` | 429 | 8 | 3 | 1 |
| `MaximalHintikka` | 158 | 6 | | 1 |
| `BinaryTreeEnumeration` | 172 | 11 | | |
| `FormulaEnumeration` | 129 | 4 | 3 | 1 |
| `ModelExistence` | 147 | 1 | 2 | 4 |
| Subtotal | 3172 | 95 | 9 | 17 |
| `PropCompactness` | 374 | 15 | | 1 |
| Total | 3546 | 110 | 9 | 18 |
| **Applications** | | | | |
| `k_coloring` | 881 | 30 | | 3 |
| `KoenigLemma` | 1966 | 66 | | 1 |
| `Hall_Theorem` | 997 | 44 | | 4 |
| `Hall_Theorem_Graphs` | 461 | 7 | | 3 |

the Lean formalization, the authors use an *inverse limit* version of the König's lemma. This lemma states that if $\{X_i\}_{i\in\mathbb{N}}$ is an indexed family of nonempty finite sets with functions $f_i : X_{i+1} \to X_i$, for each $i \in \mathbb{N}$, then there exists a family of elements $x \in \prod_i X_i$ such that $x_i = f_i(x_{i+1})$, for all $i \in \mathbb{N}$. König's lemma follows from this infinite limit version by choosing as set $X_i$ the paths of length $i$ from the root vertex $v_0$ in a tree. So, the function $f_i$ maps paths in $X_{i+1}$ into the paths without their last arc, which are paths that belong to $X_i$. The inverse limit consists of the infinite chain of functions $f_1, f_2, \ldots$. König's lemma is applied to prove the countable version of Hall's theorem by taking $M_n$ as the set of all matchings on the first $n$ indices of $I$ (i.e., the set of all possible SDRs for the sets $S_1, \ldots, S_n$), and $f_n : M_{n+1} \to M_n$ as the restriction of a match to a smaller set of indices. Since the marriage condition holds for the finite indexed families, each $M_n$ is nonempty, and by König's lemma, an element of the inverse limit gives a matching on $I$.

## 5 Conclusions and Future Work

We presented a complete formalization of the propositional Compactness theorem based on the construction of models. The Compactness theorem was applied to build full and constructive proofs of three relevant applications: Hall's theorem for countable sets and graphs, de Bruijn-Erdös theorem for countable graphs, and König's lemma.

The whole Isabelle/HOL development discussed in this paper, available through the link Compactness Theory, consists of a directory called *ModelExistence* with all required elements to prove the model existence theorem. The total number of lines in the theories related to the logical notions and properties needed on the proof of the model existence theorem is 3218, in which proofs of seventeen theorems are included (see the "subtotal" row in the Table 1). The theory *Compactness* uses the formalization of the model existence theorem and adds 15 lemmas to formalize the Compactness theorem.

Table 1 also contains information about the theories related to the discussed applications. It is remarkable to notice that the elements required to apply the Compactness theorem to prove König's lemma are almost twice the size of the other applications. Also, notice that the formalization of Hall's theorem for countable graphs is smaller since this uses directly the set-theoretical version of Hall's theorem without building any model.

As mentioned in the section on related work (Subsection 4.2), potential applications would lift combinatorial results from the infinite to the countable cases. Exploring such extensions is of remarkable interest since it is well-known that the finite cases of Hall's and de Bruijn-Erdös theorems are equivalent to other relevant combinatorial theorems.

## Data Availability Declaration

The formalizations discussed in this paper are openly available in the Isabelle *Archive of Formal Proofs* as [50] through the permanent link https://www.isa-afp.org/entries/Prop_Compactness.html

## References

1. R. Aharoni. König's Duality Theorem for Infinite Bipartite Graphs. *Journal of the London Mathematical Society*, s2-29(1), 1984. https://doi.org/10.1112/jlms/s2-29.1.1.
2. R. Aharoni, E. Berger, A. Georgakopoulos, A. Perlstein, and P. Sprüssel. The max-flow min-cut theorem for countable networks. *Journal of Combinatorial Theory Series B*, 101:1–17, 2010. https://doi.org/10.1016/j.jctb.2010.08.002.
3. R. Aharoni, M. Magidor, and R. A. Shore. On the Strength of König's Duality Theorem for Infinite Bipartite Graphs. *Journal of Combinatorial Theory Series B*, 54(2):257–290, 1992. https://doi.org/10.1016/0095-8956(92)90057-5.
4. G. Bancerek. König's Lemma. *Formalized Mathematics*, 2(3):397–402, 1991.
5. S. Berghofer. First-order logic according to fitting. *Archive of Formal Proofs*, August 2007. https://isa-afp.org/entries/FOL-Fitting.html, Formal proof development.
6. J. C. Blanchette, J. Hölzl, A. Lochbihler, L. Panny, A. Popescu, and D. Traytel. Truly Modular (Co)datatypes for Isabelle/HOL. In *5th International Conference on Interactive Theorem Proving, ITP*, volume 8558 of *LNCS*, pages 93–110. Springer, 2014. https://doi.org/10.1007/978-3-319-08970-6_7.
7. R. D. Borgersen. Equivalence of seven major theorems in combinatorics, 2004. Talk available at Department of Mathematics, University of Manitoba, Canada. https://home.cc.umanitoba.ca/~borgerse/Presentations/GS-05R-1.pdf.
8. P. J. Cameron. *Combinatorics: Topics, Techniques, Algorithms*. Cambridge University Press, 1994.
9. R. P. Dilworth. A Decomposition Theorem for Partially Ordered Sets. *Annals of Mathematics*, 51(1):161–166, 1950. https://doi.org/10.2307/1969503.
10. C. Doczkal, G. Combette, and D. Pous. A Formal Proof of the Minor-Exclusion Property for Treewidth-Two Graphs. In *Proceedings 9th International Conference on Interactive Theorem Proving - ITP*, volume 10895 of *LNCS*, pages 178–195. Springer, 2018. https://doi.org/10.1007/978-3-319-94821-8_11.
11. C. Doczkal and D. Pous. Completeness of an axiomatization of graph isomorphism via graph rewriting in Coq. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs - CPP*, pages 325–337. ACM, 2020.
12. C. Doczkal and D. Pous. Graph Theory in Coq: Minors, Treewidth, and Isomorphisms. *J. Autom. Reason.*, 64(5):795–825, 2020. https://doi.org/10.1007/s10817-020-09543-2.
13. H. B. Enderton. *A mathematical introduction to logic*. Academic Press, 1972.
14. F. Ferreira. A matemática de Kurt Gödel. Technical report, Universidade de Lisboa, 2006. In Portuguese. https://www.researchgate.net/publication/267664271_A_matematica_de_Kurt_Godel.
15. M. Fitting. *First-Order Logic and Automated Theorem Proving*. Springer-Verlag, second edition, 1996.

16. A. From. Formalizing Henkin-Style Completeness of an Axiomatic System for Propositional Logic. In *Selected Reflections in Language, Logic, and Information. ESSLLI 2019.*, LNCS, pages 80–92. Springer, 2024.
17. A. H. From, A. Schlichtkrull, and J. Villadsen. A sequent calculus for first-order logic formalized in Isabelle/HOL. *J. Log. Comput.*, 33(4):818–836, 2023.
18. G. Gonthier. A computer-checked proof of the Four Colour Theorem. Technical report, Microsoft Research, 2004.
19. G. Gonthier. The four colour theorem: Engineering of a formal proof. In D. Kapur, editor, *Computer Mathematics, 8th Asian Symposium, ASCM 2007, Singapore, December 15-17, 2007. Revised and Invited Papers*, volume 5081 of *LNCS*, page 333. Springer, 2007.
20. G. Gonthier. Formal proof – the four-color theorem. *Notices of the AMS*, 55(11):1382–1393, 2008.
21. A. Gusakov, B. Mehta, and K. A. Miller. Formalizing Hall's Marriage Theorem in Lean. *arXiv abs/2101.00127[math.CO]*, 2021. https://doi.org/10.48550/arxiv.2101.00127.
22. K. Gödel. Die Vollständigkeit der Axiome des logischen Funktionenkalküls. *Monatsh. f. Mathematik und Physik*, 37:349–360, 1930. https://doi.org/10.1007/BF01696781.
23. K. Gödel. Eine Eigenschaft der Realisierungen des Aussagenkalküls. *Ergebnisse eines mathematischen Kolloquiums*, pages 20–21, 1932. Translated as Chapter "A property of the realizationsof the propositional calculus".
24. P. Hall. On representatives of subsets. *London Mathematical Society*, 10:26–30, 1935. https://doi.org/10.1112/jlms/s1-10.37.26.
25. P. R. Halmos and H. E. Vaughan. The Marriage Problem. *American Journal of Mathematics*, 72(1):214–215, 1950.
26. J. Harrison. Formalizing Basic First Order Model Theory. In *Proceedings 11th International ConferenceTheorem Proving in Higher Order Logics TPHOL*, volume 1479 of *LNCS*, pages 153–170. Springer, 1998.
27. E. Harzheim. *Ordered Sets*, volume 7 of *Advances in Mathematics*, chapter 2. General relations between posets and their chains and antichains. Springer, 2005.
28. L. Henkin. The Completeness of the First-Order Functional Calculus. *J. Symb. Log.*, 14(3):159–166, 1949.
29. L. Henkin. Completeness in the Theory of Types. *J. Symb. Log.*, 15(2):81–91, 1950.
30. D. Jiang and T. Nipkow. Proof Pearl: The Marriage Theorem. In *Proceedings First International Conference on Certified Programs and Proofs - CPP*, volume 7086 of *LNCS*, pages 394–399, 2011. https://doi.org/10.1007/978-3-642-25379-9_28.
31. M. Kaufmann. An Extension of the Boyer-Moore Theorem Prover to Support First-Order Quantification. *J. Autom. Reason.*, 9(3):355–372, 1992.
32. H. J. Keisler. A survey of ultraproducts. In Y. Bar-Hillel, editor, *Logic, methodology and philosophy of science, Proceedings of the 1964 International Congress*, Studies in logic and the foundations of mathematics, page 112–126. North-Holland Publishing Company, 1965.
33. D. König. *Theorie Der Endlichen und Unendlichen Graphen: Kombinatorische Topologie Der Streckenkomplexe*, volume 16 of *Mathematik und ihre Anwendungen in Monographien und Lehrbüchern*. Chelsea, 1936.
34. A. Lochbihler. Coinductive. *Archive of Formal Proofs*, February 2010. https://isa-afp.org/entries/Coinductive.html, Formal proof development.
35. A. Lochbihler. A Mechanized Proof of the Max-Flow Min-Cut Theorem for Countable Networks with Applications to Probability Theory. *J. Autom. Reason.*, 66(4):585–610, 2022. https://doi.org/10.1007/s10817-022-09616-4.
36. A. I. Mal'cev. Chapter 1: Investigations in the Realm of Mathematical Logic. In *The Metamathematics of Algebraic Systems, Collected Papers: 1936–1967*, volume 66, pages 1–14. Elsevier, 1971. Translation from Russian 1936' original paper.
37. J. Michaelis and T. Nipkow. Formalized Proof Systems for Propositional Logic. In *Proceedings 23rd International Conference on Types for Proofs and Programs - TYPES 2017*, volume 104 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 5:1–5:16, 2018. https://doi.org/10.4230/LIPIcs.TYPES.2017.5.
38. L. Noschinski. Graph theory. *Archive of Formal Proofs*, April 2013. http://isa-afp.org/entries/Graph_Theory.html, Formal proof development.
39. A. C. Paseau and R. Leek. The Compactness Theorem. In *The Internet Encyclopedia of Philosophy, ISSN 2161-0002.* IEP, Visited on 24rd January, 2024. https://iep.utm.edu/wp-content/media/CompactnessTheorem.pdf.

40. B. Poizat. *A course in model theory. An introduction to contemporary mathematical logic.* Universitext. Springer, New York, Berlin, Heidelberg, 2000. English translation by Moses Klein.
41. R. Rado. Note on the transfinite case of Hall's theorem on representatives. *London Mathematical Society*, S1-42(1):321–324, 1967. https://doi.org/10.1112/jlms/s1-42.1.321.
42. P. F. Reichmeider. *The equivalence of some combinatorial matching theorems.* Polygonal Publishing House, 1985.
43. A. Robinson. *On the metamathematics of algebra.* Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1951.
44. E. Romanowicz and A. Grabowski. The Hall Marriage Theorem. *Formalized Mathematics (University of Białystok)*, 12(3):315–320, 2004. https://fm.mizar.org/2004-12/pdf12-3/hallmar1.pdf.
45. P. Rudnicki. Dilworth's Decomposition Theorem for Posets. *Formalized Mathematics*, 17(4):223–232, 2009. https://doi.org/10.2478/v10037-009-0028-4.
46. A. Schlichtkrull. *Formalization of Logic in the Isabelle Proof Assistant.* PhD thesis, Technical University of Denmark, 2018.
47. F. F. Serrano Suárez. *Formalización en Isar de la Meta-Lógica de Primer Orden.* PhD thesis, Departamento de Ciencias de la Computación e Inteligencia Artificial, Universidad de Sevilla, Spain, 2012. https://idus.us.es/handle/11441/57780. In Spanish.
48. F. F. Serrano Suárez, M. Ayala-Rincón, and T. A. de Lima. Hall's Theorem for Enumerable Families of Finite Sets. In *Proc. Int. Conf. on Intelligent Computer Mathematics CICM*, volume 13467 of *LNCS*, pages 107–121. Springer, 2022. https://doi.org/10.1007/978-3-031-16681-5_7.
49. F. F. Serrano Suárez, M. Ayala-Rincón, and T. A. de Lima. Formalisation of Hall's Theorem for Countable Infinite Graphs. In *Proc. Colombian Congress on Computation 18-CCC*, volume 2208 of *Communications in Computer and Information Science - Advances in Computing*, Cham, 2024. Springer. https://doi.org/10.1007/978-3-031-75233-9_15.
50. F. F. Serrano Suárez, T. A. de Lima, and M. Ayala-Rincón. Compactness Theorem for Propositional Logic and Combinatorial Applications. *Arch. Formal Proofs*, 2024, 2024. https://www.isa-afp.org/entries/Prop_Compactness.html.
51. N. Shankar. *Metamathematics, Machines, and Gödel's Proof.* Cambridge University Press, New York, 1994.
52. A. K. Singh. Formalization of some central theorems in combinatorics of finite sets. *arXiv abs/1703.10977[cs.Lo]*, 2017. Short presentation at the 21st International Conference on Logic for Programming, Artificial Intelligence and Reasoning - LPAR. https://doi.org/10.48550/arxiv.1703.10977.
53. A. K. Singh and R. Natarajan. Towards a Constructive Formalization of Perfect Graph Theorems. In *Proceedings 8th Indian Conference on Logic and Its Applications - ICLA*, volume 11600 of *LNCS*, pages 183–194. Springer, 2019. https://doi.org/10.1007/978-3-662-58771-3_17.
54. A. K. Singh and R. Natarajan. A constructive formalization of the weak perfect graph theorem. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs - CPP*, pages 313–324. ACM, 2020. https://doi.org/10.1145/3372885.3373819.
55. R. M. Smullyan. *First-Order Logic*, volume 43 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge.* Springer-Verlag, Berlin, 1968. Also available as a Dover Publications Inc., 1994.
56. C. Sternagel and R. Thiemann. Abstract Rewriting. *Arch. Formal Proofs*, 2010, 2010.
57. D. Traytel, A. Popescu, and J. C. Blanchette. Foundational, compositional (co)datatypes for higher-order logic: Category theory applied to theorem proving. In *Proceedings of the 27th Annual IEEE Symposium on Logic in Computer Science, LICS 2012, Dubrovnik, Croatia, June 25-28, 2012*, pages 596–605. IEEE Computer Society, 2012.