Theory and Applications of Aggregation Functions

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Overview

Motivation

- The process of combining several numerical values into a single value that somehow represents them all is called aggregation and the numerical function that carries out this process is called of aggregation function.
- When we add membership degrees or truth values in a fuzzy context (or some of its extensions) these Aggregation functions have certain properties.
- If we consider that each membership degree represents the opinion of a specialist then, if all are zeros or all are ones, the result of the aggregation must also be zero or one, respectively and furthermore, the aggregation result should be increasing.
- Other conditions can also be imposed in certain circumstances. For example commutativity.

Aggregation Functions and Properties

Definition

For each $n \geq 2$, $A : [0,1]^n \to [0,1]$ *is an n-dimensional aggregation function when* $A(0,\ldots,0) = 0$, $A(1,\ldots,1) = 1$ *and is increasing, that is,* $A(x_1,\ldots,x_n) \leq A(y_1,\ldots,y_n)$ *whenever* $x_i \leq y_i, \forall i = 1,\ldots,n$.

Definition

$\mathcal{A} : \bigcup_{n=1}^{\infty} [0,1]^n \to [0,1]$ *is an extended aggregate function if* $\mathcal{A}(x) = x$ *for each* $x \in [0,1]$ *and for any* $n \geq 2$, $\mathcal{A}|[0,1]^n$ *is an n-dimensional aggregation function.*

- Given an extended aggregation function $\mathcal{A}$, it is not necessary to have any relationship between $\mathcal{A}|[0,1]^m$ and $\mathcal{A}|[0,1]^n$ when $m \neq n$.

Extended Aggregation Functions

- The Arithmetic Mean

$$M(x_1, \ldots, x_n) = \frac{\sum_{i=1}^{n} x_i}{n}$$

for each $n \in \{1, \ldots, n\}$, is an example of extended aggregation function where there is a strong relation between the aggregation functions which form the family.

- The arithmetic mean satisfy the self-identity property:

$$M(x_1, \ldots, x_n) = M(x_1, \ldots, x_n, M(x_1, \ldots, x_n)) \tag{1}$$

Generation of extended aggregation functions

- Given a bivariate aggregation function A we can obtain an extended aggregation function $\mathcal{A} : \bigcup_{i=1}^{\infty} [0,1]^n \to [0,1]$ as following:
  1. $\mathcal{A}(x) = x$;
  2. $\mathcal{A}(x_1, \ldots, x_n) = A(\mathcal{A}(x_1, \ldots, x_{n-1}), x_n)$ for each $n \geq 2$.
- Note that if A is idempotent, i.e. $A(x,x) = x$ for all $x \in [0,1]$, then $\mathcal{A}$ satisfies the self-identity condition.
- As extended aggregation functions are generally families of independent aggregation functions, here we will only consider fixed arity aggregation functions.

Extra Properties

From now on, A will always represent an arbitrary n-dimensional aggregation function unless we explicitly say otherwise.

Idempotecy

- A is idempotent when worth $A((x)^n) = x$, $\forall x \in [0,1]$.

Proposition

A is idempotent iff $\forall \mathbf{x} \in [0,1]^n$, we have that $\min \mathbf{x} \leq A(\mathbf{x}) \leq \max \mathbf{x}$.

Proof: ($\Rightarrow$) Since A increasing and idempotent then,

$$\min \mathbf{x} = A((\min \mathbf{x})^n) \leq A(\mathbf{x}) \leq A((\max \mathbf{x})^n) = \max \mathbf{x}.$$

($\Leftarrow$) Since $\min \leq A \leq \max$, then

$$x = \min(x)^n \leq A((x)^n) \leq \max(x)^n = x.$$

So, $A((x)^n) = x$. □

Idempotency

Example

The arithmetic mean, median, minimum and maximum are idempotent aggregation functions. Two other examples of n-dimensional aggregation functions are:

$$G(x_1, \ldots, x_n) = \sqrt[n]{\prod_{i=1}^{n} x_i}$$

and for any $r > 0$*,*

$$P_{[r]}(x_1, \ldots, x_n) = \sqrt[r]{\frac{\sum_{i=1}^{n} x_i^r}{n}}$$

called of *geometric mean* *and* *power mean**, respectively.*

Internal

- A is interna if $A(\mathbf{x}) \in \{x_1, \dots, x_n\}$ for each $\mathbf{x} = (x_1, \dots, x_n) \in [0,1]^n$. .
- Clearly, every internal aggregation function also is idempotent.

Example

- *Projections, minimum, maximum and median (when n is odd) are the best known examples of internal aggregation functions.*
- *The following function is also an internal aggregation function*

$$A_s(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} = (0)^n \\ \pi_{s(\mathbf{x})}(\mathbf{x}) & \text{otherwise} \end{cases} \quad \text{where } s(\mathbf{x}) = \left\lceil \sum_{i=1}^{n} x_i \right\rceil$$

- *Let* $\lambda \in [0,1]$ *and an internal aggregation function A. Then, the function*

$$A_\lambda(\mathbf{x}) = \begin{cases} \min(\mathbf{x}) & \text{if } A(\mathbf{x}) \leq \lambda \\ \max(\mathbf{x}) & \text{if } A(\mathbf{x}) > \lambda \end{cases}$$

is also an internal aggregation function.

Commutativity

- Intuitively, A is commutative if the result does not depend on the order of the arguments.
- A is commutative if for every permutation $(p(1), \ldots, p(n))$ of $(1, \ldots, n)$ and $(x_1, \ldots, x_n) \in [0,1]^n$, we have $A(x_1, \ldots, x_n) = A(x_{p(1)}, \ldots, x_{p(n)})$.

Example

- *The arithmetic mean, the product ($P(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i$), the geometric mean, the minimum, and the maximum are examples of commutative aggregation functions.*
- *Given $i \in \{1, \ldots, n\}$, the projection π_i and the function $A(x, y) = x^y$ with the convention that $0^0 = 0$, are examples of non-commutative aggregation functions.*

Commutativity and ordering

- Notation: For each $\mathbf{x} \in [0,1]^n$, $\mathbf{x}_\nearrow$ and $\mathbf{x}_\searrow$ denote the increasing and decreasing ordering of $\mathbf{x}$.

Proposition

Let A be an n-dimensional aggregation function. Then the functions $A_\nearrow, A_\searrow : [0,1]^n \to [0,1]$ *defined by*

$$A_\nearrow(\mathbf{x}) = A(\mathbf{x}_\nearrow) \text{ and } A_\searrow(\mathbf{x}) = A(\mathbf{x}_\searrow)$$

are commutative n-dimensional aggregation functions.

Obviously, when A is commutative then $A = A_\nearrow = A_\searrow$.

Neutral Element

- An element $e \in [0,1]$ is a neutral element of A if for all $x \in [0,1]$, $A(e,\dots,e,x,e,\dots,e) = x$.
- Not every aggregation function has a neutral element (for example the arithmetic mean). But, if A has a neutral element then this one is unique.

Example

Let A be an n-dimensional aggregation function such that $\min \leq A \leq \max$. *So given any $e \in [0,1]$ fixed the function*

$$A_e(\mathbf{x}) = \begin{cases} \min(\mathbf{x}) & \textit{if } \mathbf{x} \leq (e)^n \\ \max(\mathbf{x}) & \textit{if } \mathbf{x} \geq (e)^n \\ A(\mathbf{x}) & \textit{else} \end{cases}$$

is an idempotent aggregation function with e as neutral element.

Neutral Element

Proposition

Let A_1 and A_2 be n-dimensional aggregation functions with neutral elements e_1 and e_2, respectively. If $A_1 \leq A_2$ then $e_2 \leq e_1$.

Proof: suppose that $e_1 < e_2$. Then, $e_2 = A_1(e_2, e_1, \ldots, e_1) \leq A_2(e_2, e_1, \ldots, e_1) \leq A_2(e_2, \ldots, e_2, e_1) = e_1$ which is an contradiction. Therefore, $e_2 \leq e_1$. □

Corollary

If $A \leq \min$ and A has a neutral element e, then $e = 1$.

Corollary

If $A \geq \max$ and A has a neutral element a, then a is 0.

Absorbing element

- $a \in [0,1]$ is an absorbing element of A, if for each $x_1, \ldots, x_{n-1} \in [0,1]$ and $i \in \{1, \ldots, n\}$, we have that $A(x_1, \ldots, x_{i-1}, a, x_i, \ldots, x_{n-1}) = a$.
- Clearly, if A has an absorbing element, then it is unique.

Proposition

Let A_1 and A_2 be n-dimensional aggregation function with absorbing elements a_1 and a_2, respectively. If $A_1 \leq A_2$ then $a_1 \leq a_2$.

Proof: $a_1 = A_1(a_1, \ldots, a_1, a_2) \leq A_2(a_1, \ldots, a_1, a_2) = a_2$. □

Corollary

If $A \leq \min$ then 0 is the absorbing element of A.

Proof: Direct from above Proposition, because 0 is the absorbing element of the $\min$. □

Absorbing element

Corollary

If $A \geq \max$ then 1 is the absorbing element of A.

- Note that there are aggregation functions with 0 as an absorbing element that are greater than the minimum. For example, the geometric mean.
- Similarly, there are aggregation functions that have 1 as an absorbing element that are smaller than the maximum. For example, given an aggregation function A such that $A \leq \max$, then the function:

$$A_A(\mathbf{x}) = \begin{cases} 1 & \text{if } \max(\mathbf{x}) = 1 \\ A(\mathbf{x}) & \text{otherwise} \end{cases}$$

has 1 as the absorbing element and is smaller than max.

Absorbing element

Example

For each $a \in [0,1]$ *the function*

$$A_a(\mathbf{x}) = \begin{cases} 0 & \textit{if } \mathbf{x} = (0)^n \\ 1 & \textit{if } \mathbf{x} = (1)^n \\ a & \textit{otherwise} \end{cases}$$

is an n-dimensional aggregation function with a as absorbing element.

Proposition

If A has an absorbing element $a \in (0,1)$ *then A has no neutral element.*

Proof: Let e be a neutral element of A. If $a < e$ we have that $a = A(0, a \ldots, a) \leq A(0, e, \ldots, e) = 0$ and therefore $a = 0$ which is a contradiction. In the case of $e \leq a$ we have that $1 = A(1, e, \ldots, e) \leq A(1, a, \ldots, a) = a$ and therefore $a = 1$ which is a contradiction. □

Linear aggregation functions

- A is shift-invariant if $\forall\lambda \in [-1, 1]$

$$A(\lambda + x_1, \ldots, \lambda + x_n) = A(x_1, \ldots, x_n) + \lambda$$

  whenever $x_i \in [0, 1]$, $x_i + \lambda \in [0, 1]$ and $A(x_1, \ldots, x_n) + \lambda \in [0, 1]$ with $i = 1, \ldots, n$.

- A is homogeneous if for each $\lambda \in [0, 1]$

$$A(\lambda x_1, \ldots, \lambda x_n) = \lambda A(x_1, \ldots, x_n)$$

  whenever $x_i \in [0, 1]$ with $i = 1, \ldots, n$.

- An aggregation function which is both, shift-invariant and homogeneous is called of linear.
- All linear aggregation function is idempotent.
- The minimum, maximum, projections and arithmetic mean are examples of linear aggregation functions.

Classification of Aggregation Functions

There are several classifications of aggregation functions, but the best known and accepted one is the one that considers the minimum and maximum as referents to obtain four classes of aggregation functions and, except for the minimum and maximum itself, every aggregation function belongs to one and just one of these classes.

- Averaging;
- Conjunctives;
- Disjunctives;
- Mixed.

Averaging functions

- An aggregation function A is considered a averaging function if $\min \leq A \leq \max$.
- The averaging result cannot exceed the highest value nor be below the lowest value of the entries. This is important if we want to fusion the opinion of several experts on how much a given alternative satisfies a given criterion.

Proposition

A is an averaging function iff A is idempotent.

Averaging functions based on weighting vectors

- In some situations it is necessary to consider that the inputs have different weights. This is the case, for example, when in decision making the assessment of a senior specialist on whether a given alternative meets a criterion has greater weight than that of another specialist with less qualifications.
- A vector $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$ is called of weighting vector if $\sum_{i=1}^{n} w_i = 1$.
- Listed below are some examples of averaging functions based on an arbitrary weighting vector $\mathbf{w}$. Consider $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$
  - Weighted arithmetic mean: $M_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$.
  - Ordered weighted averaging (OWA):
    $owa_{\mathbf{w}}(\mathbf{x}) = M_{\mathbf{w}_\searrow}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_{(i)}$ with $(x_{(1)}, \ldots, x_{(n)}) = \mathbf{x}_\searrow$.

Examples of averaging functions based on weighting vectors

- Reverse ordered weighted averaging (ROWA):
  $Rowa_{\mathbf{w}}(\mathbf{x}) = M_{\mathbf{w}\nearrow}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_{(i)}$ com $(x_{(1)}, \ldots, x_{(n)}) = \mathbf{x}_{\nearrow}$.

- Weighted geometric mean: $G_{\mathbf{w}}(\mathbf{x}) = \prod_{i=1}^{n} x_i^{w_i}$.

- Weighted power mean: $P_{\mathbf{w},[r]}(\mathbf{x}) = \left(\sum_{i=1}^{n} w_i x_i^r\right)^{\frac{1}{r}}$ for a fixed $r > 0$.

- Weighted harmonic mean: $H_{\mathbf{w}}(\mathbf{x}) = \left(\sum_{i=1}^{n} \frac{w_i}{x_i}\right)^{-1}$ with the convention that $\frac{w_i}{0} = 0$.

- Sine trigonometric mean: $SM_{\mathbf{w}}(\mathbf{x}) = \frac{2}{\pi} \arcsin\left(\sum_{i=1}^{n} w_i \sin(\frac{\pi}{2} x_i)\right)$.

OWA-like averaging functions

- OWA is not a specific aggregation function, but rather a family of them.
- Every OWA is a ROWA and vice versa.
- Every weighted averaging function supports an OWA-like version.
- For example,
  - The ordered geometric mean is the following:

$$OWG_{\mathbf{w}}(\mathbf{x}) = G_{\mathbf{w}}(\mathbf{x}_{\searrow}) = \prod_{i=1}^{n} x_{(i)}^{w_i}$$

  - Let $r > 0$. The ordered weighted power mean is the following:

$$OWP_{\mathbf{w},[r]}(\mathbf{x}) = P_{\mathbf{w},[r]}(\mathbf{x}_{\searrow}) = \left(\sum_{i=1}^{n} w_i x_{(i)}^{r}\right)^{\frac{1}{r}}$$

Choquet integrals

- Choquet integrals are a widely used family of aggregation functions that generalize the Lebsegue integral.
- It was introduced in 1953 by the French mathematician Gustave Choquet.
- The main characteristic of Choquet integrals is that they allow the inputs to be aggregated in a way that not only considers the importance of individual inputs, as is the case with weighted averages, nor their magnitude, as is the case with averages weighted orders, but also takes groups (or coalitions) into account.
- Choquet integrals are based on measurements.

Fuzzy measures

Definition

*A **fuzzy measure** is a function* $m : \wp(\{1, \ldots, n\}) \to [0, 1]$ *such that for all* $X, Y \subseteq \{1, \ldots, n\}$ *we have that*

1. *If* $X \subseteq Y$ *then* $m(X) \leq m(Y)$*;*
2. $m(\emptyset) = 0$ *and* $m(\mathbb{N}_n) = 1$*.*

- Fuzzy measures theory considers a generalization of the notion of measure where the additive property is replaced by a weak monotonicity property.
- The best known fuzzy measures are:
  - Uniform measure: $m_U(X) = \frac{|X|}{n}$
  - Dirac Measure w.r.t. $i \in \{1, \ldots, n\}$:

$$m^i_D(X) = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{if } i \notin X \end{cases}$$

Fuzzy measures examples

- additive measure w.r.t. a weighting vector $\omega = (w_1, \ldots, w_n)$:

$$m^{\omega}(X) = \sum_{i \in X} w_i.$$

- Symmetric measure w.r.t. a weighting vector $\omega = (w_1, \ldots, w_n)$:

$$m_s^{\omega}(X) = \sum_{i=1}^{|X|} w_i.$$

- Power measure w.r.t. $p > 0$: $m_P^p(X) = \left(\frac{|X|}{n}\right).$
- Bedregal relative measure: $m_R(X) = \frac{\sum_{i \in X} i}{\sum_{i=1}^{n} i} = \frac{2 \sum_{i \in X} i}{n(n+1)}.$

Discrete Choquet integrals

Definition

Let $m : \wp(\{1, \dots, n\}) \to [0,1]$ be a fuzzy measure. The ***discrete Choquet integral*** *w.r.t. m is the function $C_m : [0,1]^n \to [0,1]$ defined by*

$$C_m(\mathbf{x}) = \sum_{i=1}^{n} (x_{(i)} - x_{(i-1)}) m(\Gamma_i) \tag{2}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in [0,1]^n$ and $(x_{(1)}, \dots, x_{(n)})$ is a permutation increasing of $\mathbf{x}$, that is, a permutation such that $0 = x_{(0)} \leq x_{(1)} \leq \dots \leq x_{(n)}$ and $\Gamma_i = \{(i), \dots, (n)\}$.

Proposition

Given a fuzzy measure $m : \wp(\{1, \dots, n\}) \to [0,1]$ the discrete Choquet integral w.r.t. m is a linear and idempotent aggregation function.

Theorem

An aggregation function $A : [0,1]^n \to [0,1]$ *is a Choquet integral, i.e.* $A = C_m$ *for some fuzzy measure m if and only if A is linear and additive comonotone*

- The OWA's (and therefore the minimum, maximum, and the weighted arithmetic mean) are Choquet integrals.
- Recently some generalizations of these integrals have been proposed.

Conjunctives Aggregation Functions

- Conjunctive aggregation unctions model the fuzzy conjunction, that is, the logical "and" in fuzzy logic.
- An n-dimensional aggregation function A is **conjunctive** if $A \leq \min$.
- Therefore, every conjunctive aggregation function has 0 as an absorbing element and if it has a neutral element, this is necessarily 1.
- In fact, every aggregation function with 1 as a neutral element is conjunctive, but not every conjunctive aggregation function has a neutral element, such as the function

$$A(\mathbf{x}) = \min(x_1^2, \ldots, x_n^2).$$

- Those aggregation functions that have 1 as a neutral element are called **semi-copulas**.

Triangular norms

- Triangular norms were introduced in 1942 by Menger to generalize the axiom of triangular inequality of (metric) distances in metric spaces probabilistic.
- The notion of triangular norm given by Menger was very general, as it did not require the existence of a fixed neutral element nor associativity.
- Later, Schweizer and Sklar divided the triangular Norms into t-norms and t-conorms (the dual operation to t-norm), and in the case of t-norms they treat them as a semigroup operation on the unitary interval $[0, 1]$ with neutral element 1.
- Alsina, Trillas and Valverde in 1980 used t-norms to model fuzzy conjunction, generalizing different interpretations to fuzzy conjunction proposals until then.

T-norms

Definition

*$T : [0,1] \times [0,1] \to [0,1]$ is a triangular norm (**t-norm**) if it satisfies the following axioms:*

(T1) $T(x, y) = T(y, x)$ *– Commutativity;*

(T2) $T(x, T(y, z)) = T(T(x, y), z)$ *– Associativity;*

(T3) *If $x \leq x'$ and $y \leq y'$ then $T(x, y) \leq T(x', y')$ – Monotonicity;*

(T4) $T(x, 1) = x$ *– 1-Identity.*

- Note that for any t-norm T, since $T(0, 1) = 0$, and that by monotonicity $T(0, 0) \leq T(0, 1)$, then $T(0, x) = 0$ and by the commutativity $T(x, 0) = 0$ for all $x \in [0, 1]$.
- Therefore, $T(x, y) = x \wedge y$ when $x, y \in \{0, 1\}$, that is, every t-norm behaves like the classical conjunction when we only consider Boolean values 0 and 1.

Examplos of t-norms

- Gödel: $T_G(x, y) = \min(x, y)$
- Łukasiewicz: $T_L(x, y) = \max(x + y - 1, 0)$
- Product: $T_P(x, y) = xy$
- Weak:

$$T_W(x, y) = \begin{cases} 0 & \text{if } \max(x, y) < 1 \\ T_G(x, y) & \text{otherwise} \end{cases}$$

- Hamacher for $\gamma \geq 0$: $T_{H,\gamma}(x, y) = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)}$

Disjunctive Aggregation Functions

- Disjunctive aggregation functions model fuzzy disjunction or the “or” of fuzzy logic.
- An aggregation function A is disjunctive if $\max \leq A$.
- There is a duality between disjunctive and conjunctive aggregation functions. Given a conjunctive aggregation function $A : [0,1]^n \to [0,1]$, the function $A^d : [0,1]^n \to [0,1]$, defined by :

$$A^d(x_1, \ldots, x_n) = 1 - A(1 - x_1, \ldots, 1 - x_n)$$

  is a disjunctive aggregation function.
- The duals of t-norms are called t-norms. Basically, a t-conorm is an aggregation function of arity 2, which is commutative, associative and has 0 as a neutral element.

Mixed Aggregation Functions

- Mixed aggregation functions are aggregation functions that are neither average, neither conjunctive nor disjunctive.
- An n-dimensional aggregate function A is mixed if there exists $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ such that either $A(\mathbf{x}) < \min \mathbf{x}$ and $\min \mathbf{y} < A(\mathbf{y})$, or, $A(\mathbf{x}) < \max \mathbf{x}$ and $\max \mathbf{y} < A(\mathbf{y})$. Therefore, mixed aggregation functions are not comparable with the minimum or are not comparable with the maximum.
- Mixed aggregation functions are useful in decision making when some attributes are positive while others negative.
- For example, some symptoms can contribute to closing the diagnosis of a patient while others may contribute to ruling out this diagnosis.

Uninorms

- An important class of mixed aggregation functions is the class of uninorms which are neither t-norms nor t-connorms.
- Uninorms are a generalization of the notion of t-norm and t-conorm.
- An aggregation function $U : [0,1]^2 \to [0,1]$ is a uninorm if it is commutative, associative and has a neutral element.

Proposition

Let U be a uninorm. U is a mixed aggregation function if and only if the neutral element of U is different from 0 and 1.

Proposition

Let $U : [0,1]^2 \to [0,1]$ *be an uninorm with* $e \in [0,1]$ *as neutral element. If* $e > \min(x,y)$ *and* $\max(x,y) > e$ *then* $\min(x,y) \leq U(x,y) \leq \max(x,y)$.

Aplications in Decision Making

- There are several methods in which fuzzy logic can assist in decision making.
- Most of them use aggregation functions
- These methods can be classified into two:
  - Based on fuzzy and
  - Based on fuzzy decision matrices

Decision making based on fuzzy decision matrices

- A fuzzy decision matrix is a $[0,1]$-valued matrix where the rows represent the alternatives, the columns the criteria or attributes considered, and the values in each position (i,j) represent how much the i-th alternative meets the j-th attribute.
- Elements of multi-attribute and multi-expert decision making based on decision matrices:
  - A set $E = \{e_1, \ldots, e_m\}$ of experts
  - A set $X = \{x_1, \ldots, x_n\}$ of alternatives
  - A set $A = \{a_1, \ldots, a_p\}$ of attributes or criteria
  - A vector of weights $\mathbf{w} = (w_1, \ldots, w_p)^T$ for the attributes
  - Decision matrices $R^1, \ldots, R^m$ of dimension $n \times p$

Proposed method for decision making based on fuzzy decision matrices

Step 1: Aggregate the matrices R^1 to R^m into a single matrix $\mathcal{RC}$, called of consensus decision matrix, defined for each $i = 1, \ldots, n$ and $j = 1, \ldots, p$, as:

$$\mathcal{RC}_{ij} = owa_\Lambda(R^1_{ij}, \ldots, R^m_{ij}) \quad (3)$$

where $\Lambda = (\lambda_1, \ldots, \lambda_m)$ is the weight vector:

- If m is even: $\lambda_i = \frac{1}{2^{\frac{m}{2}+2-i}} + \frac{1}{2^{\frac{m}{2}} m}$ for every $i = 1, \ldots, \frac{m}{2}$, and $\lambda_i = \lambda_{m+1-i}$ for every $i = \frac{m}{2} + 1, \ldots, m$.
- If m is odd: $\lambda_i = \frac{1}{2^{\frac{m+1}{2}+2-i}} + \frac{1}{2^{\frac{m+1}{2}} m} + \frac{1}{4m}$ for every $i = 1, \ldots, \frac{m+1}{2}$, and $\lambda_i = \lambda_{m+1-i}$ for every $i = \frac{m+1}{2} + 1, \ldots, m$.

Proposed method for decision making based on fuzzy decision matrices

Step 2: For each alternative x_i, with $i = 1, \ldots, n$, using the weighted arithmetic mean, determine the total collective index O_i as follows:

$$O_i = M_{\mathbf{w}}(\mathcal{RC}_{i1}, \ldots, \mathcal{RC}_{in}) \quad (4)$$

Step 3: Determine a ranking of alternatives based on the total collective indices in such a way that $x_i > x_j$ when $O_i > O_j$ and $x_i \sim x_j$ when $O_i = O_j$.

Illustrative example

Consider the problem of purchasing a central air conditioning system. Suppose that we have

- Three alternative of central air conditioning systems : $\{A_1, A_2, A_3\}$;
- Four attributes: a_1 (economy), a_2 (functionality), a_3 (operability) and a_4 (longevity);
- The vector of weights for the attributes is: $\mathbf{w} = (0.2134, 0.1707, 0.2805, 0.3354)$.
- three experts: $\{p_1, p_2, p_3\}$;
- Each expert provides a grade to represent how much, in his opinion, each one of the three alternatives satisfy each of the four criteria. These expert assessments are placed in the form of Tables, as for example:

Illustrative Example: Expert Decision Matrices

Tabela: Decision matrix of p_1

R^1	a_1	a_2	a_3	a_4
A_1	0.6	0.45	0.45	0.35
A_2	0.6	0.4	0.55	0.15
A_3	0.6	0.7	0.55	0.7

Table: Decision matrix of p_2

R^2	a_1	a_2	a_3	a_4
A_1	0.7	0.45	0.65	0.55
A_2	0.75	0.55	0.65	0.55
A_3	0.55	0.65	0.6	0.4

Tabela: Decision matrix of p_3.

R^3	a_1	a_2	a_3	a_4
A_1	0.6	0.35	0.55	0.45
A_2	0.55	0.55	0.5	0.4
A_3	0.4	0.4	0.45	0.55

Illustrative Example: Consensus Decision Matrix

As there are three experts ($m = 3$), the weighting vector is calculated as follows:

$$\lambda_1 = \frac{1}{2^3} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3} = \frac{1}{8} + \frac{1}{6} = 0.291\overline{6}$$
$$\lambda_2 = \frac{1}{2^2} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3} = \frac{1}{4} + \frac{1}{6} = 0.41\overline{6}$$
$$\lambda_3 = \frac{1}{2^3} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3} = \frac{1}{8} + \frac{1}{6} = 0.291\overline{6}.$$

Determine the consensus decision matrix RC obtained from Tables R^1, R^2 and R^3 considering Equation (3). For example, RC_{11} is calculated as follows:

$$\begin{aligned} RC_{11} &= owa_{\Lambda}(R^1_{11}, R^2_{11}, R^3_{11}) \\ &= owa_{\Lambda}(0.6, 0.7, 0.6) \\ &= 0.291\overline{6} \cdot 0.6 + 0.41\overline{6} \cdot 0.6 + 0.291\overline{6} \cdot 0.7 \\ &= 0.6291\overline{6} \end{aligned}$$

Illustrative Example: Consensus Decision Matrix

Tabela: Consensus decision matrix

RC	a_1	a_2	a_3	a_4
A_1	$0.6291\overline{6}$	$0.4208\overline{3}$	0.55	0.45
A_2	$0.6291\overline{6}$	0.50625	$0.56458\overline{3}$	$0.3708\overline{3}$
A_3	$0.5208\overline{3}$	$0.591\overline{6}$	$0.53541\overline{6}$	0.55

Illustrative Example: Total collective index and final ranking

The total collective index is calculated using Equation (**??**) as follows:

- $O_1 = 0.521671\overline{6}$
- $O_2 = 0,517220 41\overline{6}$
- $O_3 = 0,54226208\overline{3}$

For example, below show how O_1 was obtained.

$$\begin{aligned} O_1 &= M_{\mathbf{w}}(0.6291\overline{6}, 0.4208\overline{3}, 0.55, 0, 45) \\ &= 0.2134 \cdot 0.6291\overline{6} + 0.1707 \cdot 0.4208\overline{3} + 0.2805 \cdot 0.55 + 0.3354 \cdot 0.45 \\ &= 0.51130541\overline{6} \end{aligned}$$

So, from this total collective index, we obtain the following ranking of alternatives:

$$A_3 > A_1 > A_2$$

The Image Reduction using Aggregation Functions

The image reduction process consists basically in a mechanism that reduces the size of as we illustrate below.

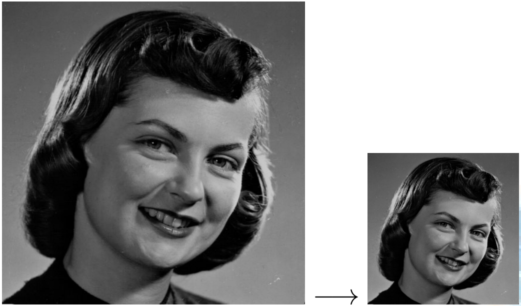

Figura: Illustration of the image reduction process.

The Image Reduction using Aggregation Functions

The greatest motivation to think about reducing an image is the possibility of storing the same information using a smaller memory space than the original one.

The problem is that by reducing an image, we lose some of the original information, since there is no way to get the original image from the reduced image.

The Image Reduction using Aggregation Functions

Definition:

An image in grayscale can be defined as a matrix M of size $n \times m$, with elements $M(i, j)$ in the continuous interval $[0, 1]$. Each element $M(i, j)$ of the matrix M represents one pixel.

Remark:

- A color image is obtained by combining three grayscale layers.
- For simplicity, we will only use grayscale images.

The Image Reduction using **GM** Functions

In the figure below we see an example of 6*x*8 grayscale image.

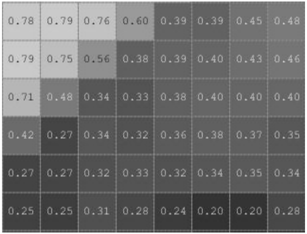

Figura: Representation of an image 6×8 in grayscale.

The Image Reduction using Aggregation Functions

Among the many techniques of image reduction, we describe the simple method used by Farias et al.[1]:

Image reduction algorithm:

To reduce an image we follow these steps:

1. Input a $n \times m$ image;
2. Input the scale of reduction, i.e., input natural numbers n_0 and m_0 such that $\frac{n}{n_0}$ and $\frac{m}{m_0}$ are natural numbers;
3. Partition the image into distinct blocks of size $n_0 \times m_0$;
4. Choose an aggregation function $A : [0,1]^{n_0 \cdot m_0} \longrightarrow [0,1]$;
5. Apply the selected function in each block;
6. Build the new image;

[1] A.D.S. Farias, V.S. Costa, L.R.A. Lopes, R.H.N. Santiago, B. Bedregal. **On Generalized Mixture Functions**. Transactions on Fuzzy Sets and Systems, Vol.1, No.2, (2022), 99–128.

The Image Reduction using aggregation Functions

In the figure below we illustrate the image reduction process.

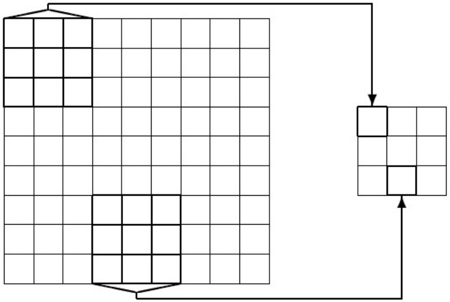

Figura: Example of image reduction by blocks 3×3.

Other Applications

- Classifier ensembles take a collection of classifiers that process the same information and their output is combined in some way. In the following paper, this combination is made with a generalization of the Choquet integral:

  Thiago V. V. Batista, Benjamín R. C. Bedregal, Ronei M. Moraes: Constructing multi-layer classifier ensembles using the Choquet integral based on overlap and quasi-overlap functions. Neurocomputing 500: 413-421 (2022)

- Applications of pseudo overlap (grouping) functions in multi-attribute (group) decision-making,fuzzy mathematical morphology and image processing are discussed in the following paper:

  Xiaohong Zhang, Rong Liang, Humberto Bustince, Benjamín R. C. Bedregal, Javier Fernández, Mengyuan Li, Qiqi Ou: Pseudo Overlap Functions, Fuzzy Implications and Pseudo Grouping Functions with Applications. Axioms 11(11): 593 (2022)

Other Applications

- A class of extended aggregation function was used in a wavelet-fuzzy power quality diagnosis method which was evaluated an experimental microgrid with different energy sources and load types.

  D.H.S. Nolasco, F.B. Costa, E.S. Palmeira, D.K. Alves, B. Bedregal, T.O.A. Rocha, R.L.A. Ribeiro, J.C.L. Silva. Wavelet–fuzzy power quality diagnosis system with inference method based on overlap functions: Case study in an AC microgrid. Eng. Appl. Artif. Intell. 85: 284–294 (2019)

- An accurate way to tackle classification problems is by using fuzzy rule-based classification systems (FRBCS) and using a generalization of Choquet integrals, we obtain an statistical tie with the state of the art in FRBCS.

  G. Lucca, G.P. Dimuro, J. Fernández, H. Bustince, B.. Bedregal, J.A. Sanz. Improving the Performance of Fuzzy Rule-Based Classification Systems Based on a Nonaveraging Generalization of CC-Integrals Named CF1F2-Integrals. IEEE Trans. Fuzzy Syst. 27(1): 124–134 (2019)


THANKS!