

# Closed Rewriting

## Checking overlaps of Nominal Rewriting rules

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# Overview

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2. Main Problem
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## Concepts and Definitions

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# Nominal syntax

Nominal Signature  $\Sigma$ : set of *function symbols*  $f, g, \wedge, \exists, \dots$

Meta-level unknowns  $\mathcal{X}$ : set of *variables*  $X, Y, P, Q, \dots$

Object-level variables  $\mathcal{A}$ : set of atoms  $a, b, c, \dots$

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## Nominal terms

*Nominal terms* are generated inductively by the grammar:

$$t := a \quad | \quad \pi \cdot X \quad | \quad [a]t \quad | \quad f(t_1, \dots, t_n)$$

$\Sigma$ ,  $\mathcal{X}$  and  $\mathcal{A}$  are pairwise disjoint.

# Permutation and Substitution

**Permutation  $\pi$ :** is a bijection on atoms, with finite domain.

A **swapping  $(a\ b)$**  is a pair of atoms that maps  $a$  to  $b$ ,  $b$  to  $a$  and all other atoms  $c$  to themselves.

$$(a\ b) \cdot a = b \quad (a\ b)(b\ c) \cdot a = b$$

$$(a\ b)(b\ c) \cdot b = c \quad (a\ b)(b\ c) \cdot c = a$$

**Substitution  $\theta$ :** is a mapping from a finite set of variables to terms.

$$\theta = [X \mapsto P, Y \mapsto \forall[a]Q]$$

$$(X \wedge Y)\theta = P \wedge \forall[a]Q$$

# Constraints

Freshness constraints (denoted by #): Intuitively,  $a\#t$  means that  $a$  does not occur free in  $t$  (read “ $a$  fresh in  $t$ ”).

$$a\#b \quad a\#a \quad a\#[a]a$$

$\alpha$ -equivalence constraints (denoted by  $\approx_\alpha$ ): Intuitively,  $s \approx_\alpha t$  means that  $s$  and  $t$  are  $\alpha$ -equivalent, that is, they are the same term written with a different choice of bound names.

$$\lambda x.x \approx_\alpha \lambda y.y \quad u\lambda x.x \not\approx_\alpha v\lambda y.y \quad \lambda z.zy \approx_\alpha \lambda x.xy$$

# Nominal Commutative Unification

A **problem**  $Pr$  is defined as a set of constraints of the form  $a\#X$  and  $s \approx_{\alpha, \mathcal{C}} t$ .

## Definition

A **C-solution** for a triple  $\mathcal{P} = (\Delta, \delta, Pr)$  is a pair  $(\Delta', \theta)$  where the following conditions are satisfied:

1.  $\Delta' \vdash \Delta\theta$ ;
2.  $\Delta' \vdash a\#t\theta$ , if  $a\#t \in Pr$ ;
3.  $\Delta' \vdash s\theta \approx_{\alpha, \mathcal{C}} t\theta$ , if  $s \approx_{\alpha, \mathcal{C}} t \in Pr$ ;
4. there is a substitution  $\theta'$  such that  $\Delta' \vdash \delta\theta' \approx_{\alpha, \mathcal{C}} \theta$ .

If there is no  $(\Delta', \theta)$  then we say that the problem  $\mathcal{P}$  is *unsolvable*. Also  $\mathcal{U}_{\mathcal{C}}(\mathcal{P})$  denotes the set of all C-solutions of the triple  $\mathcal{P}$ .



# Nominal Commutative Unification

## Definition

A *nominal C-unification problem (in context)* is a pair of the form  $(\nabla \vdash l) \stackrel{\mathbf{C}}{? \approx ?} (\Delta \vdash s)$ .

The pair  $(\Delta', \theta)$  is a **C-solution** of  $(\nabla \vdash l) \stackrel{\mathbf{C}}{? \approx ?} (\Delta \vdash s)$  iff  $(\Delta', \theta)$  is a **C-solution** of the triple  $\mathcal{P} = (\{\nabla, \Delta\}, \text{Id}, \{l \approx_{\alpha, \mathbf{C}} s\})$ .

- ⊙  $\mathcal{U}_{\mathbf{C}}(\nabla \vdash l, \Delta \vdash s)$  denotes the set of all **C-solutions** of  $(\nabla \vdash l) \stackrel{\mathbf{C}}{? \approx ?} (\Delta \vdash s)$ .

**C-solutions** are found using a sound and complete (not finitary) rule-based algorithm for **C-unification** [AdCSFN17].

# Nominal Commutative Matching

## Definition

A *nominal C-matching problem* is a pair of terms-in-context  $(\nabla \vdash l) \stackrel{C}{\approx} (\Delta \vdash s)$  where  $V(\nabla \vdash l) \cap V(\Delta \vdash s) = \emptyset$ .

A *C-solution* to this problem is a substitution  $\theta$  such that

1.  $\Delta \vdash \nabla\theta$ ;
2.  $\Delta \vdash l\theta \approx_{\alpha, C} s$  and
3.  $\text{dom}(\theta) \subseteq V(\nabla \vdash l)$ .

# Nominal Rewriting modulo C

## Nominal rewriting modulo C:

The *one-step rewrite modulo C relation*  $\Delta \vdash s \rightarrow_{R,C} t$  is the least relation such that for any  $R = (\nabla \vdash l \rightarrow r) \in R$ , position  $C$ , term  $s'$ , permutation  $\pi$ , and substitution  $\theta$ ,

$$\frac{s \equiv C[s'] \quad \Delta \vdash (\nabla \theta, s' \approx_{\alpha,C} \pi \cdot (l\theta), C[\pi \cdot (r\theta)] \approx_{\alpha,C} t)}{\Delta \vdash s \rightarrow_{R,C} t}$$

# Nominal Rewriting modulo C

Example (Nominal rules for prenex normal form):

Consider  $\Sigma = \{\forall, \exists, \neg, \wedge, \vee\}$  the signature for first-order logic.

Let  $C = \{\vdash P \vee Q \approx Q \vee P, \vdash P \wedge Q \approx Q \wedge P\}$  be a set of identities.

Let  $C$  be the theory over  $\Sigma$  consisting of the following rules:

$$R_1 : a\#P \vdash P \wedge \forall[a]Q \rightarrow \forall[a](P \wedge Q)$$

$$R_2 : a\#P \vdash P \vee \forall[a]Q \rightarrow \forall[a](P \vee Q)$$

$$R_3 : a\#P \vdash P \wedge \exists[a]Q \rightarrow \exists[a](P \wedge Q)$$

$$R_4 : a\#P \vdash P \vee \exists[a]Q \rightarrow \exists[a](P \vee Q)$$

$$R_5 : \vdash \neg(\exists[a]Q) \rightarrow \forall[a]\neg Q$$

$$R_6 : \vdash \neg(\forall[a]Q) \rightarrow \exists[a]\neg Q$$

$$R_7 : \vdash \exists[a](\forall[b]Q) \rightarrow \forall[b](\exists[a]Q)$$

# Nominal Rewriting modulo C

$$a\#P' \vdash S' \vee (P' \vee \exists[a]Q')$$

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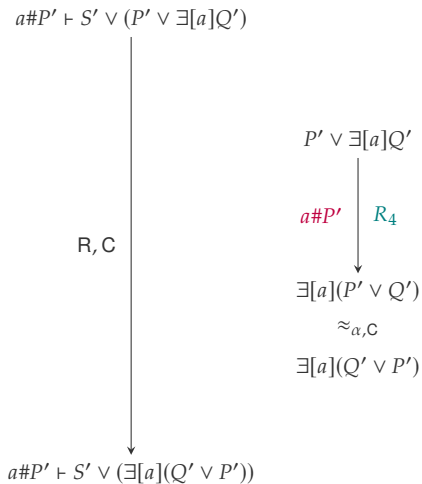
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$$a\#P' \vdash S' \vee (\exists[a](Q' \vee P'))$$
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$a\#P'$   $R_4$

$$\exists[a](P' \vee Q')$$

# Nominal Rewriting modulo C





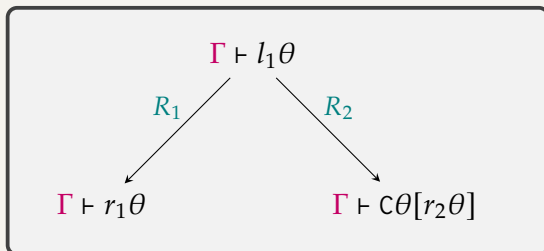
# Nominal Rewriting modulo C

- ⊙ If  $\Delta \vdash s \rightarrow_{R,C}^* t$  and  $\Delta \vdash s \rightarrow_{R,C}^* t'$ , then we say a nominal rewrite system R is *C-confluent* when there exists a term  $u$  such that  $\Delta \vdash t \rightarrow_{R,C}^* u$  and  $\Delta \vdash t' \rightarrow_{R,C}^* u$ .
- ⊙ R is said to be *C-terminating* if there is no infinite rewrite modulo C sequence.
- ⊙ R is *C-convergent* if it is C-confluent and C-terminating.

# Critical Pairs

## (Overlaps and critical pairs)

We say  $R_1 = \nabla_1 \vdash l_1 \rightarrow r_1$  *overlaps* with  $R_2 = \nabla_2 \vdash l_2 \rightarrow r_2$ , and we call then the pair of terms-in-context  $\Gamma \vdash \langle r_1\theta, C\theta[r_2\theta] \rangle$  a *critical pair*,



whenever  $l_1 \equiv C[l'_1]$  such that  $\{\nabla_1, \nabla_2, l'_1 \approx? l_2\}$  has a principal solution  $(\Gamma, \theta)$ , so that  $\Gamma \vdash l'_1\theta \approx_\alpha l_2\theta$  and  $\Gamma \vdash \nabla_i\theta$  for  $i = 1, 2$ .

# Nominal Equality

An *equational theory*  $E = (\Sigma, Ax)$  is a pair of a signature  $\Sigma$  and a possibly infinite set of equality judgements  $Ax$  in  $\Sigma$ , called *axioms*.

## (Nominal algebra) equality

*(Nominal algebra) equality*:  $\Delta \vdash_E s = t$  is the least transitive reflexive symmetric relation such that for any  $(\nabla \vdash l = r) \in E$ , position  $C$ , permutation  $\pi$ , substitution  $\theta$ , and fresh  $\Gamma$  (so if  $a\#X \in \Gamma$  then  $a$  is not mentioned in  $\Delta, s, t$ ),

$$\frac{\Delta, \Gamma \vdash (\nabla \theta, \quad s \approx_\alpha C[\pi \cdot (l\theta)], \quad C[\pi \cdot (r\theta)] \approx_\alpha t)}{\Delta \vdash_E s = t}$$

## Mismatch - Nominal Algebra and Nominal Rewriting

In general, nominal rewriting is not complete for equational reasoning. We just saw that nominal algebra includes an extra fresh context  $\Gamma$ , which does not match the rewriting reasoning.

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Spoiler alert: closed nominal rewriting is complete! [FG10]

Main Problem

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# Main Problem

**Example:** Consider  $\Sigma = \{\forall, \exists, \neg, \wedge, \vee\}$  the signature for first-order logic.

Let  $\mathbf{C} = \{\vdash P \vee Q \approx Q \vee P, \vdash P \wedge Q \approx Q \wedge P\}$  be a set of identities.

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$$R_7 : \vdash \exists[a](\forall[b]Q) \rightarrow \forall[b](\exists[a]Q)$$

# Main Problem

Critical pair:  $R_3$  versus  $R_7$ .

$$R_3 : a_3\#P_3 \vdash P_3 \wedge \exists[a_3]Q_3 \rightarrow \exists[a_3](P_3 \wedge Q_3)$$

$$R_7 : \vdash \exists[a_7](\forall[b_7]Q_7) \rightarrow \forall[b_7](\exists[a_7]Q_7)$$

We solve the nominal C-unification problem (in-context):

$$(a_3\#P_3 \vdash \exists[a_3]Q_3) \text{ ?}\approx\text{?} (\emptyset \vdash \exists[a_7](\forall[b_7]Q_7))$$

and get the C-solution:

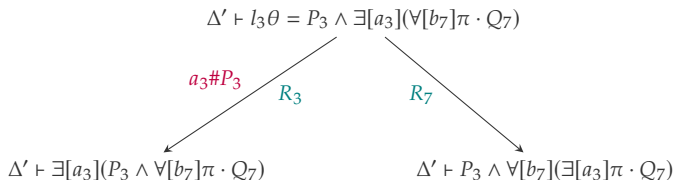
$$(\Delta' = \{a_3\#P_3, a_3\#Q_7\}, \theta = [Q_3 \mapsto \forall[b_7](a_3 a_7) \cdot Q_7])$$



# Main Problem

Let  $\Delta' = \{a_3\#P_3, a_3\#Q_7\}$  and  $\pi = (a_3 \ a_7)$ . We get the following critical pair (diagram below):

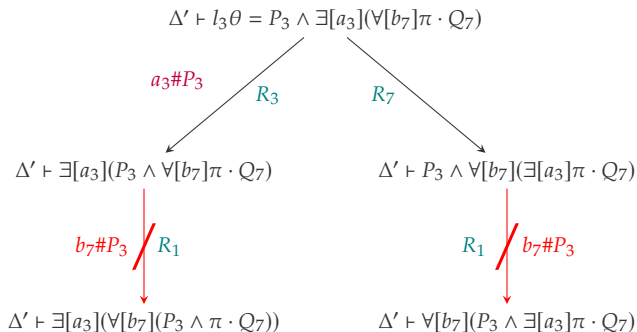
$$\Delta' \vdash \langle \exists[a_3](P_3 \wedge \forall[b_7]\pi \cdot Q_7), P_3 \wedge \forall[b_7](\exists[a_3]\pi \cdot Q_7) \rangle$$



# Main Problem

$$\Delta' = \{a_3\#P_3, a_3\#Q_7\}$$

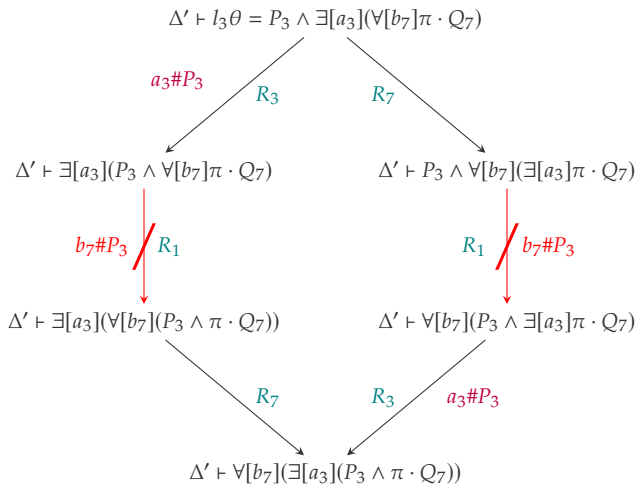
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## Main Problem

Problem: Note that we could only make the reduction in red if we had  $b_7 \# P_3 \in \Delta'$ .

Notice that  $b_7$  is a new name that was chosen to rename the Rule 7. And we could have chosen a  $b_7$  that is fresh in  $P_3$ .

It seems that we need to weaken the context with new names fresh for the variables occurring in the rules.

Here we need closedness. [FG10]

## Closedness

Intuitively, no free atom occurs in a closed term – closed axioms do not allow abstracted atoms to become free.

If  $t$  is a term, we say that  $t^n$  is a **freshened variant** of  $t$  when  $t^n$  has the same structure as  $t$ , except that the atoms and unknowns have been replaced by ‘fresh’ atoms and unknowns.

$$[a][b]X : \quad [a^n][b^n]X^n \quad [a^n][a^n]X^n \quad [a^n][b^n]X$$

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## Closed term (in-context)

A term-in-context  $\nabla \vdash l$  is **closed** if there exists a solution for the matching problem

$$(\nabla^n \vdash l^n) \stackrel{?}{\approx} (\nabla, A(\nabla^n, l^n) \# V(\nabla, l) \vdash l).$$

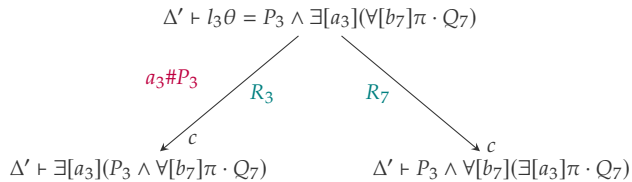
## Extending results

### Closed Nominal Rewriting modulo C

The *one-step closed rewrite modulo C relation*  $\Delta \vdash s \xrightarrow{R,C}_c t$  is the least relation such that for any  $R = (\nabla \vdash l \rightarrow r) \in \mathbf{R}$  and term-in-context  $\Delta \vdash s$ , there is some  $R^n$  a freshened variant of  $R$  (so fresh for  $R, \Delta, s, t$ ), position  $C$ , term  $s'$ , permutation  $\pi$ , and substitution  $\theta$ ,

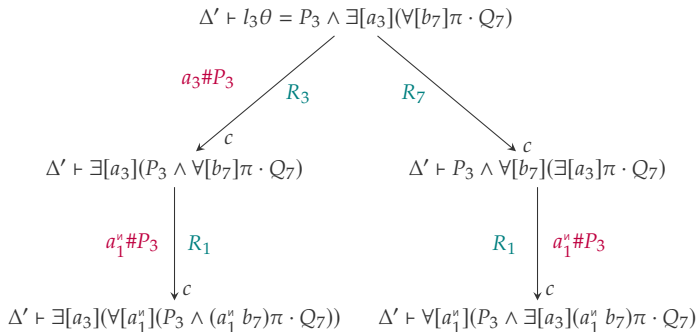
$$\frac{s \equiv C[s'] \quad \Delta, A(R^n)\#V(\Delta, s, t) \vdash (\nabla^n\theta, s' \approx_{\alpha,C} l^n\theta, C[r^n\theta] \approx_{\alpha,C} t)}{\Delta \vdash s \xrightarrow{R,C}_c t}$$

# Problem fixed

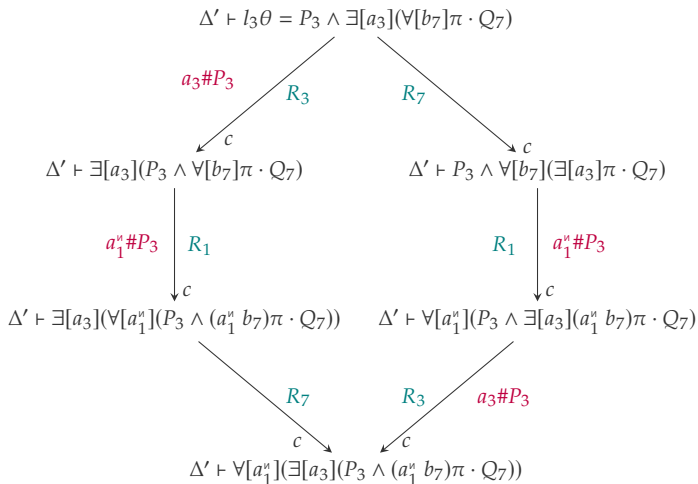




# Problem fixed



# Problem fixed



## Conclusion and Future Work

- ⊙ Closedness is essential to guarantee the confluence of this particular NRS – it simplifies the computation of critical pairs.
- ⊙ A nominal critical pair modulo  $C$  is a new concept that is under investigation:
  - we still need to prove a version of the nominal Critical Pair Lemma modulo  $C$ .
- ⊙ We want to apply the current extensions in the development of closed nominal narrowing modulo  $C$ :
  - we have to prove a version of the nominal Lifting Theorem modulo  $C$ .

## References

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END

# Appendix

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# Simplification rules for C-unification

We follow the approach by Ayala et. al. [AdCSFN17].

|         |  |
|---------|--|
| (#ab)   | $(\Delta, \theta, Pr \uplus \{a\#b\}) \implies (\Delta, \theta, Pr)$   |
| (#app)  | $(\Delta, \theta, Pr \uplus \{a\#f(t_1, \dots, t_n)\}) \implies (\Delta, \theta, Pr \cup \{a\#t_1, \dots, a\#t_n\})$ |
| (#a[a]) | $(\Delta, \theta, Pr \uplus \{a\#[a]t\}) \implies (\Delta, \theta, Pr)$  |
| (#a[b]) | $(\Delta, \theta, Pr \uplus \{a\#[b]t\}) \implies (\Delta, \theta, Pr \cup \{a\#t\})$                                |
| (#var)  | $(\Delta, \theta, Pr \uplus \{a\#\pi \cdot X\}) \implies (\{(\pi^{-1} \cdot a)\#X\} \cup \Delta, \theta, Pr)$        |

|                                      |  |
|--------------------------------------|--|
| $(\approx_{\alpha, C} \text{ refl})$ | $(\Delta, \theta, Pr \uplus \{s \approx_{\alpha, C} s\}) \implies (\Delta, \theta, Pr)$  |
| $(\approx_{\alpha, C} \text{ app})$  | $(\Delta, \theta, Pr \uplus \{f(\bar{s})_n \approx_{\alpha, C} f(\bar{t})_n\}) \implies (\Delta, \theta, Pr \cup \bigcup \{s_i \approx_{\alpha, C} t_i\})$   |
| $(\approx_{\alpha, C} C)$            | $(\Delta, \theta, Pr \uplus \{f^C s \approx_{\alpha, C} f^C t\}) \implies (\Delta, \theta, Pr \cup \{s \approx_{\alpha, C} v\})$ , where $s = (s_0, s_1)$<br>and $t = (t_0, t_1), v = (t_i, t_{(i+1) \bmod 2}), i = 0, 1$  |
| $(\approx_{\alpha, C} \text{ [aa]})$ | $(\Delta, \theta, Pr \uplus \{[a]s \approx_{\alpha, C} [a]t\}) \implies (\Delta, \theta, Pr \cup \{s \approx_{\alpha, C} t\})$   |
| $(\approx_{\alpha, C} \text{ [ab]})$ | $(\Delta, \theta, Pr \uplus \{[a]s \approx_{\alpha, C} [b]t\}) \implies (\Delta, \theta, Pr \cup \{s \approx_{\alpha, C} (a b) \cdot t, a\#t\})$   |
| $(\approx_{\alpha, C} \text{ inst})$ | $(\Delta, \theta, Pr \uplus \{\pi \cdot X \approx_{\alpha, C} t\}) \implies (\Delta, \theta', Pr[X \mapsto \pi^{-1} \cdot t] \cup \bigcup_{\substack{Y \in \text{dom}(\theta'), \\ a\#Y \in \Delta}} \{a\#Y\theta'\})$<br>let $\theta' := \theta[X \mapsto \pi^{-1} \cdot t]$ ,<br>if $X \notin \text{Var}(t)$ |
| $(\approx_{\alpha, C} \text{ inv})$  | $(\Delta, \theta, Pr \uplus \{\pi \cdot X \approx_{\alpha, C} \pi' \cdot X\}) \implies (\Delta, \theta, Pr \cup \{\pi \oplus (\pi')^{-1} \cdot X \approx_{\alpha, C} X\})$<br>if $\pi' \neq \text{Id}$   |



# Nominal Rewriting

## Nominal rewriting

The *one-step rewrite relation*  $\Delta \vdash s \xrightarrow{R}_{[C,R,\theta,\pi]} t$  is the least relation such that for any  $R = (\nabla \vdash l \rightarrow r) \in \mathbb{R}$ , position  $C$ , term  $s'$ , permutation  $\pi$ , and substitution  $\theta$ ,

$$\frac{s \equiv C[s'] \quad \Delta \vdash (\nabla \theta, s' \approx_{\alpha} \pi \cdot (l\theta), C[\pi \cdot (r\theta)] \approx_{\alpha} t)}{\Delta \vdash s \xrightarrow{R}_{[C,R,\theta,\pi]} t}$$

- ⊙ To find  $\theta$  and  $\pi$  above, we need to solve the nominal matching problem  $(\Delta \vdash s') \approx? (\nabla \vdash l)$ .

# Nominal Rewriting

- ⊙ A NRS is said to be *confluent* when for all  $\Delta, s, t$  and  $t'$  such that  $\Delta \vdash s \rightarrow^* t$  and  $\Delta \vdash s \rightarrow^* t'$ , there exists  $u$  such that  $\Delta \vdash t \rightarrow^* u$  and  $\Delta \vdash t' \rightarrow^* u$ .

Notice we need the same  $\Delta$  here. We will find some complications later.

# Nominal Rewriting

Since atoms are not affected by substitution actions but can be swapped, we need to consider a technicality called *equivariance*.

- ⊙ The *equivariant closure* of a set  $Rw$  of rewrite rules is the closure of  $Rw$  by the meta-action of permutations, that is, it is the set of all permutative variants of rules in  $Rw$ . We denote *eq-closure*( $Rw$ ) for the equivariant closure of  $Rw$ .

## Nominal Rewriting

Consider the NRS with the single rule  $R \equiv a\#X \vdash f(X, b) \rightarrow a$ . In order to find the *eq-closure*( $Rw$ ), we need to analyze all the permutative variants of  $R \in Rw$ , they are  $R^{(a\ b)}$ ,  $R^{(a\ c)}$ ,  $R^{(b\ c)}$  and  $R^{(a\ c)(b\ d)}$ , where  $c, d$  are arbitrary new atoms.

$$R_1 = R^{(a\ b)} = b\#X \vdash f(X, a) \rightarrow b$$

$$R_2 = R^{(a\ c)} = c\#X \vdash f(X, b) \rightarrow c$$

$$R_3 = R^{(b\ c)} = a\#X \vdash f(X, c) \rightarrow a$$

$$R_4 = R^{(a\ c)(b\ d)} = c\#X \vdash f(X, d) \rightarrow c$$

Therefore, *eq-closure*( $Rw$ ) =  $\{R, R_1, R_2, R_3, R_4\}$ .

# Critical Pairs

## (Permutative overlaps and critical pairs)

Let  $R_1 = \nabla_1 \vdash l_1 \rightarrow r_1$  and  $R_2 = \nabla_2 \vdash l_2 \rightarrow r_2$  be copies of two rewrite rules in  $eq\text{-closure}(Rw)$  such that there is an overlap.

If  $R_2$  is a copy of  $R_1^\pi$ , we say that the overlap is *permutative*.

A permutative overlap at the root position is called *root-permutative*.

We call an overlap that is not trivial and not root-permutative *proper*.

The same terminology is used to classify critical pairs.

# Critical Pairs

## (Peak and local confluence)

Let  $R$  be an equivariant rewrite system, and let  $\Delta, s, t_1$  and  $t_2$  such that  $\Delta \vdash s \rightarrow t_1$  and  $\Delta \vdash s \rightarrow t_2$ . This pair will be denoted as  $\Delta \vdash s \rightarrow t_1, t_2$  and called a *peak*.

If there is such a peak, then we call a NRS *locally confluent* when there exists a term  $u$  such that  $\Delta \vdash t_1 \rightarrow^* u$  and  $\Delta \vdash t_2 \rightarrow^* u$ . We say such a peak is *joinable*.

Notice we need the same  $\Delta$  here again.

In this way, we can only say that a critical pair is joinable if its terms are under the same context.

# Main Problem

Let  $\Delta = \{a_3 \# P_3\}$ .

$$\begin{aligned} & (\Delta, \emptyset, \{l_3 |_2 \text{ ?} \approx \text{ ?} l_7\}) = \\ & = (\Delta, \emptyset, \{\exists[a_3]Q_3 \text{ ?} \approx \text{ ?} \exists[a_7](\forall[b_7]Q_7)\}) \\ & \Rightarrow_{(\approx_{\alpha, C\text{app}})} (\Delta, \emptyset, \{[a_3]Q_3 \text{ ?} \approx \text{ ?} [a_7](\forall[b_7]Q_7)\}) \\ & \Rightarrow_{(\approx_{\alpha, C[ab]})} (\Delta, \emptyset, \{Q_3 \text{ ?} \approx \text{ ?} (a_3 \ a_7) \cdot \forall[b_7]Q_7, a_3 \# \forall[b_7]Q_7\}) \\ & \Rightarrow_{(\#_{\text{app}})}^2 (\Delta, \emptyset, \{Q_3 \text{ ?} \approx \text{ ?} \forall[b_7](a_3 \ a_7) \cdot Q_7, a_3 \# Q_7\}) \\ & \Rightarrow_{(\approx_{\alpha, C\text{inst}})} (\Delta, \theta = [Q_3 \mapsto \forall[b_7](a_3 \ a_7) \cdot Q_7], \\ & \quad \{\forall[b_7](a_3 \ a_7) \cdot Q_7 \text{ ?} \approx \text{ ?} \forall[b_7](a_3 \ a_7) \cdot Q_7, a_3 \# Q_7\}) \\ & \Rightarrow_{(\approx_{\alpha, C\text{refl}})} (\Delta, \theta, \{a_3 \# Q_7\}) \\ & \Rightarrow_{(\#_{\text{var}})} (\Delta \cup \{a_3 \# Q_7\}, \theta, \emptyset) \end{aligned}$$

## Nominal rewriting not complete for equational reasoning

Suppose  $R$  is a presentation of  $E$ . It is **not** necessarily the case that

$$\Delta \vdash_E s = t \quad \text{implies} \quad \Delta \vdash_R s \leftrightarrow t.$$

Take  $E = \{a\#X \vdash X = f(X)\}$  and  $R = \{a\#X \vdash X \rightarrow f(X)\}$ .

Then we have  $\vdash_E X = f(X)$  by definition, using  $\Gamma = a\#X$ , but  $\not\vdash_R X \leftrightarrow f(X)$ .



# Nominal Narrowing [AFN16]

## Nominal Narrowing

The *one-step narrowing relation*  $(\Delta \vdash s) \rightsquigarrow_{[C,R,\theta,\pi]} (\Delta' \vdash t)$  is the least relation such that for any  $R = (\nabla \vdash l \rightarrow r) \in R$ , position  $C$ , term  $s'$ , permutation  $\pi$ , and substitution  $\theta$ ,

$$\frac{s \equiv C[s'] \quad \Delta' \vdash (\nabla\theta, \Delta\theta, s'\theta \approx_\alpha \pi \cdot (l\theta), (C[\pi \cdot r])\theta \approx_\alpha t)}{(\Delta \vdash s) \rightsquigarrow_{[C,R,\theta,\pi]} (\Delta' \vdash t)}$$

- ⊙ To find  $\theta$  and  $\pi$  above, we need to solve the nominal unification problem  $(\Delta \vdash s') \stackrel{?}{\approx} (\nabla \vdash l)$ .

# Definition closedness

## Closed rewriting

The *one-step closed rewrite relation*  $\Delta \vdash s \xrightarrow{R}_c t$  is the least relation such that for any  $R = (\nabla \vdash l \rightarrow r) \in R$  and term-in-context  $\Delta \vdash s$ , there is some  $R^n$  a freshened variant of  $R$  (so fresh for  $R, \Delta, s, t$ ), position  $C$ , term  $s'$ , permutation  $\pi$ , and substitution  $\theta$ ,

$$\frac{s \equiv C[s'] \quad \Delta, A(R^n) \# V(\Delta, s, t) \vdash (\nabla^n \theta, s' \approx_\alpha l^n \theta, C[r^n \theta] \approx_\alpha t)}{\Delta \vdash s \xrightarrow{R}_c t}$$