Closed Rewriting

Checking overlaps of Nominal Rewriting rules

Daniella Santaguida Magalhães de Souza
Advisor: Daniele Nantes Sobrinho

Date: 2024/02/08

Overview

Concepts and Definitions

Nominal syntax

Nominal Signature Σ: set of *function symbols* $f, g, \wedge, \exists, \ldots$

Meta-level unknowns $\mathfrak{X}$: set of *variables* $X, Y, P, Q, \ldots$

Object-level variables $\mathscr{A}$: set of atoms $a, b, c, \ldots$

Nominal syntax

Nominal Signature Σ: set of *function symbols* $f, g, \wedge, \exists, \ldots$

Meta-level unknowns $\mathcal{X}$: set of *variables* $X, Y, P, Q, \ldots$

Object-level variables $\mathcal{A}$: set of atoms $a, b, c, \ldots$

Nominal terms

Nominal terms are generated inductively by the grammar:

$$t := a \quad | \quad \pi \cdot X \quad | \quad [a]t \quad | \quad f(t_1, \ldots, t_n)$$

Σ, $\mathcal{X}$ and $\mathcal{A}$ are pairwise disjoint.

Permutation and Substitution

Permutation π: is a bijection on atoms, with finite domain.

A swapping $(a\ b)$ is a pair of atoms that maps a to b, b to a and all other atoms c to themselves.

$$(a\ b) \cdot a = b \qquad (a\ b)(b\ c) \cdot a = b$$

$$(a\ b)(b\ c) \cdot b = c \quad (a\ b)(b\ c) \cdot c = a$$

Substitution θ: is a mapping from a finite set of variables to terms.

$$\theta = [X \mapsto P, Y \mapsto \forall[a]Q]$$

$$(X \wedge Y)\theta = P \wedge \forall[a]Q$$

Constraints

Freshness constraints (denoted by #): Intuitively, $a\#t$ means that a does not occur free in t (read "a fresh in t").

$$a\#b \qquad a \not\# a \qquad a\#[a]a$$

α-equivalence constraints (denoted by $\approx_\alpha$): Intuitively, $s \approx_\alpha t$ means that s and t are α-equivalent, that is, they are the same term written with a different choice of bound names.

$$\lambda x.x \approx_\alpha \lambda y.y \qquad u\lambda x.x \not\approx_\alpha v\lambda y.y \qquad \lambda z.zy \approx_\alpha \lambda x.xy$$

Nominal Commutative Unification

A problem Pr is defined as a set of constraints of the form $a\#X$ and $s \approx_{\alpha,\mathsf{C}} t$.

Definition

A C-*solution* for a triple $\mathcal{P} = (\Delta, \delta, Pr)$ is a pair (Δ', θ) where the following conditions are satisfied:

1. $\Delta' \vdash \Delta\theta$;
2. $\Delta' \vdash a\#t\theta$, if $a\#t \in Pr$;
3. $\Delta' \vdash s\theta \approx_{\alpha,\mathsf{C}} t\theta$, if $s \approx_{\alpha,\mathsf{C}} t \in Pr$;
4. there is a substitution θ' such that $\Delta' \vdash \delta\theta' \approx_{\alpha,\mathsf{C}} \theta$.

If there is no (Δ', θ) then we say that the problem $\mathcal{P}$ is *unsolvable*. Also $\mathcal{U}_{\mathsf{C}}(\mathcal{P})$ denotes the set of all C-solutions of the triple $\mathcal{P}$.

Nominal Commutative Unification

Definition

A *nominal* C*-unification problem (in context)* is a pair of the form $(\nabla \vdash l) \; {}_?\overset{\mathsf{C}}{\approx}_? \; (\Delta \vdash s)$.

The pair (Δ', θ) is a C-solution of $(\nabla \vdash l) \; {}_?\overset{\mathsf{C}}{\approx}_? \; (\Delta \vdash s)$ iff (Δ', θ) is a C-solution of the triple $\mathscr{P} = (\{\nabla, \Delta\}, \mathtt{Id}, \{l \approx_{\alpha,\mathsf{C}} s\})$.

- $\mathscr{U}_{\mathsf{C}}(\nabla \vdash l, \Delta \vdash s)$ denotes the set of all C-solutions of $(\nabla \vdash l) \; {}_?\overset{\mathsf{C}}{\approx}_? \; (\Delta \vdash s)$.

C-solutions are found using a sound and complete (not finitary) rule-based algorithm for C-unification [AdCSFN17].

Nominal Commutative Matching

Definition

A *nominal* C*-matching problem* is a pair of terms-in-context $(\nabla \vdash l) \; {}_{?}\overset{\mathsf{C}}{\approx} (\Delta \vdash s)$ where $V(\nabla \vdash l) \cap V(\Delta \vdash s) = \emptyset$.

A C*-solution* to this problem is a substitution θ such that

1. $\Delta \vdash \nabla\theta$;
2. $\Delta \vdash l\theta \approx_{\alpha,\mathsf{C}} s$ and
3. $dom(\theta) \subseteq V(\nabla \vdash l)$.

Nominal Rewriting modulo C

Nominal rewriting modulo C:

The *one-step rewrite modulo* C *relation* $\Delta \vdash s \rightarrow_{\mathsf{R},\mathsf{C}} t$ is the least relation such that for any $R = (\nabla \vdash l \rightarrow r) \in \mathsf{R}$, position $\mathtt{C}$, term s', permutation π, and substitution θ,

$$\frac{s \equiv \mathtt{C}[s'] \qquad \Delta \vdash \left(\nabla\theta,\ s' \approx_{\alpha,\mathsf{C}} \pi \cdot (l\theta),\ \mathtt{C}[\pi \cdot (r\theta)] \approx_{\alpha,\mathsf{C}} t\right)}{\Delta \vdash s \rightarrow_{\mathsf{R},\mathsf{C}} t}$$

Nominal Rewriting modulo C

Example (Nominal rules for prenex normal form):

Consider $\Sigma = \{\forall, \exists, \neg, \wedge, \vee\}$ the signature for first-order logic.

Let $\mathsf{C} = \{ \vdash P \vee Q \approx Q \vee P,\ \vdash P \wedge Q \approx Q \wedge P\}$ be a set of identities.

Let C be the theory over Σ consisting of the following rules:

$$
\begin{array}{llll}
R_1: & a\#P & \vdash & P \wedge \forall[a]Q \longrightarrow \forall[a](P \wedge Q) \\
R_2: & a\#P & \vdash & P \vee \forall[a]Q \longrightarrow \forall[a](P \vee Q) \\
R_3: & a\#P & \vdash & P \wedge \exists[a]Q \longrightarrow \exists[a](P \wedge Q) \\
R_4: & a\#P & \vdash & P \vee \exists[a]Q \longrightarrow \exists[a](P \vee Q) \\
R_5: & & \vdash & \neg(\exists[a]Q) \longrightarrow \forall[a]\neg Q \\
R_6: & & \vdash & \neg(\forall[a]Q) \longrightarrow \exists[a]\neg Q \\
R_7: & & \vdash & \exists[a](\forall[b]Q) \longrightarrow \forall[b](\exists[a]Q)
\end{array}
$$

Nominal Rewriting modulo C

$$a\#P' \vdash S' \vee (P' \vee \exists[a]Q')$$

Nominal Rewriting modulo C

$$
\begin{tikzcd}
a\#P' \vdash S' \vee (P' \vee \exists[a]Q') \arrow[d, "\mathsf{R},\mathsf{C}"'] \\
a\#P' \vdash S' \vee (\exists[a](Q' \vee P'))
\end{tikzcd}
$$

Nominal Rewriting modulo C

$$
\begin{tikzcd}
a\#P' \vdash S' \vee (P' \vee \exists[a]Q') \arrow[d, "\mathsf{R,C}"'] \\
a\#P' \vdash S' \vee (\exists[a](Q' \vee P'))
\end{tikzcd}
\qquad
\begin{tikzcd}
P' \vee \exists[a]Q' \arrow[d, "a\#P'"', "R_4"] \\
\exists[a](P' \vee Q')
\end{tikzcd}
$$

Nominal Rewriting modulo C

$$\begin{tikzcd}
a\#P' \vdash S' \vee (P' \vee \exists[a]Q') \arrow[dddd, "\mathsf{R},\mathsf{C}"'] & \\
& P' \vee \exists[a]Q' \arrow[d, "a\#P'"', "R_4"] \\
& \exists[a](P' \vee Q') \\
& \approx_{\alpha,\mathsf{C}} \\
& \exists[a](Q' \vee P') \\
a\#P' \vdash S' \vee (\exists[a](Q' \vee P')) &
\end{tikzcd}$$

Nominal Rewriting modulo C

- If $\Delta \vdash s \rightarrow^*_{\mathsf{R},\mathsf{C}} t$ and $\Delta \vdash s \rightarrow^*_{\mathsf{R},\mathsf{C}} t'$, then we say a nominal rewrite system R is C-*confluent* when there exists a term u such that $\Delta \vdash t \rightarrow^*_{\mathsf{R},\mathsf{C}} u$ and $\Delta \vdash t' \rightarrow^*_{\mathsf{R},\mathsf{C}} u$.
- R is said to be C-*terminating* if there is no infinite rewrite modulo C sequence.
- R is C-*convergent* if it is C-confluent and C-terminating.

Critical Pairs

(Overlaps and critical pairs)

We say $R_1 = \nabla_1 \vdash l_1 \longrightarrow r_1$ *overlaps* with $R_2 = \nabla_2 \vdash l_2 \longrightarrow r_2$, and we call then the pair of terms-in-context $\Gamma \vdash \langle r_1\theta, \mathtt{C}\theta[r_2\theta]\rangle$ a *critical pair*,

$$\begin{tikzcd}
 & \Gamma \vdash l_1\theta \arrow[dl, "R_1"'] \arrow[dr, "R_2"] & \\
\Gamma \vdash r_1\theta & & \Gamma \vdash \mathtt{C}\theta[r_2\theta]
\end{tikzcd}$$

whenever $l_1 \equiv \mathtt{C}[l_1']$ such that $\{\nabla_1, \nabla_2, l_1' \; {}_?{\approx}_? \; l_2\}$ has a principal solution (Γ, θ), so that $\Gamma \vdash l_1'\theta \approx_\alpha l_2\theta$ and $\Gamma \vdash \nabla_i\theta$ for $i = 1, 2$.

Nominal Equality

An *equational theory* $\mathsf{E} = (\Sigma, Ax)$ is a pair of a signature Σ and a possibly infinite set of equality judgements Ax in Σ, called *axioms*.

(Nominal algebra) equality

(Nominal algebra) equality: $\Delta \vdash_{\mathsf{E}} s = t$ is the least transitive reflexive symmetric relation such that for any $(\nabla \vdash l = r) \in \mathsf{E}$, position $\mathtt{C}$, permutation π, substitution θ, and fresh Γ (so if $a\#X \in \Gamma$ then a is not mentioned in Δ, s, t),

$$\frac{\Delta, \Gamma \vdash \left(\nabla\theta, \quad s \approx_\alpha \mathtt{C}[\pi \cdot (l\theta)], \quad \mathtt{C}[\pi \cdot (r\theta)] \approx_\alpha t\right)}{\Delta \vdash_{\mathsf{E}} s = t}$$

Mismatch - Nominal Algebra and Nominal Rewriting

In general, nominal rewriting is not complete for equational reasoning. We just saw that nominal algebra includes an extra fresh context Γ, which does not match the rewriting reasoning.

Mismatch - Nominal Algebra and Nominal Rewriting

In general, nominal rewriting is not complete for equational reasoning. We just saw that nominal algebra includes an extra fresh context Γ, which does not match the rewriting reasoning.

Spoiler alert: closed nominal rewriting is complete! [FG10]

Main Problem

Main Problem

Example: Consider $\Sigma = \{\forall, \exists, \neg, \wedge, \vee\}$ the signature for first-order logic.

Let $\mathsf{C} = \{\vdash P \vee Q \approx Q \vee P,\ \vdash P \wedge Q \approx Q \wedge P\}$ be a set of identities.

Let C be the theory over Σ consisting of the following rules:

$$
\begin{array}{lll}
R_1: & a\#P & \vdash P \wedge \forall[a]Q \longrightarrow \forall[a](P \wedge Q) \\
R_2: & a\#P & \vdash P \vee \forall[a]Q \longrightarrow \forall[a](P \vee Q) \\
R_3: & a\#P & \vdash P \wedge \exists[a]Q \longrightarrow \exists[a](P \wedge Q) \\
R_4: & a\#P & \vdash P \vee \exists[a]Q \longrightarrow \exists[a](P \vee Q) \\
R_5: & & \vdash \neg(\exists[a]Q) \longrightarrow \forall[a]\neg Q \\
R_6: & & \vdash \neg(\forall[a]Q) \longrightarrow \exists[a]\neg Q \\
R_7: & & \vdash \exists[a](\forall[b]Q) \longrightarrow \forall[b](\exists[a]Q)
\end{array}
$$

Main Problem

Critical pair: R_3 versus R_7.

$$\begin{array}{llcl} R_3: & a_3\#P_3 & \vdash & P_3 \wedge \exists[a_3]Q_3 \longrightarrow \exists[a_3](P_3 \wedge Q_3) \\ R_7: & & \vdash & \exists[a_7](\forall[b_7]Q_7) \longrightarrow \forall[b_7](\exists[a_7]Q_7) \end{array}$$

We solve the nominal C-unification problem (in-context):

$$(a_3\#P_3 \vdash \exists[a_3]Q_3) \;{}_?{\approx}_?\; (\emptyset \vdash \exists[a_7](\forall[b_7]Q_7))$$

and get the C-solution:

$$(\Delta' = \{a_3\#P_3, a_3\#Q_7\}, \theta = [Q_3 \mapsto \forall[b_7](a_3\ a_7)\cdot Q_7])$$

Main Problem

Let $\Delta' = \{a_3\#P_3, a_3\#Q_7\}$ and $\pi = (a_3\ a_7)$. We get the following critical pair (diagram below):

$$\Delta' \vdash \langle \exists[a_3](P_3 \wedge \forall[b_7]\pi \cdot Q_7), P_3 \wedge \forall[b_7](\exists[a_3]\pi \cdot Q_7) \rangle$$

$$\begin{tikzcd}[column sep=small]
& \Delta' \vdash l_3\theta = P_3 \wedge \exists[a_3](\forall[b_7]\pi \cdot Q_7) \arrow[dl, "a_3\#P_3"', "R_3"] \arrow[dr, "R_7"'] & \\
\Delta' \vdash \exists[a_3](P_3 \wedge \forall[b_7]\pi \cdot Q_7) & & \Delta' \vdash P_3 \wedge \forall[b_7](\exists[a_3]\pi \cdot Q_7)
\end{tikzcd}$$

Main Problem

$\Delta' = \{a_3\#P_3, a_3\#Q_7\}$

In order to check if this critical pair is joinable, we continue:

$$
\begin{tikzcd}
& \Delta' \vdash l_3\theta = P_3 \wedge \exists[a_3](\forall[b_7]\pi \cdot Q_7) \arrow[dl, "a_3\#P_3"', "R_3"] \arrow[dr, "R_7"'] & \\
\Delta' \vdash \exists[a_3](P_3 \wedge \forall[b_7]\pi \cdot Q_7) \arrow[d, "b_7\#P_3"', "R_1", "/" marking] & & \Delta' \vdash P_3 \wedge \forall[b_7](\exists[a_3]\pi \cdot Q_7) \arrow[d, "R_1"', "b_7\#P_3", "/" marking] \\
\Delta' \vdash \exists[a_3](\forall[b_7](P_3 \wedge \pi \cdot Q_7)) & & \Delta' \vdash \forall[b_7](P_3 \wedge \exists[a_3]\pi \cdot Q_7)
\end{tikzcd}
$$

Main Problem

$$\Delta' = \{a_3\#P_3, a_3\#Q_7\}$$

In order to check if this critical pair is joinable, we continue:

$$
\begin{tikzcd}[column sep=small]
 & \Delta' \vdash l_3\theta = P_3 \wedge \exists[a_3](\forall[b_7]\pi \cdot Q_7) \arrow[dl, "R_3"', "a_3\#P_3"] \arrow[dr, "R_7"] & \\
\Delta' \vdash \exists[a_3](P_3 \wedge \forall[b_7]\pi \cdot Q_7) \arrow[d, "b_7\#P_3"', "R_1", "/" marking] & & \Delta' \vdash P_3 \wedge \forall[b_7](\exists[a_3]\pi \cdot Q_7) \arrow[d, "R_1"', "b_7\#P_3", "/" marking] \\
\Delta' \vdash \exists[a_3](\forall[b_7](P_3 \wedge \pi \cdot Q_7)) \arrow[dr, "R_7"] & & \Delta' \vdash \forall[b_7](P_3 \wedge \exists[a_3]\pi \cdot Q_7) \arrow[dl, "R_3"', "a_3\#P_3"] \\
 & \Delta' \vdash \forall[b_7](\exists[a_3](P_3 \wedge \pi \cdot Q_7)) &
\end{tikzcd}
$$

Main Problem

Problem: Note that we could only make the reduction in red if we had $b_7 \# P_3 \in \Delta'$.

Notice that b_7 is a new name that was chosen to rename the Rule 7. And we could have chosen a b_7 that is fresh in P_3.

It seems that we need to weaken the context with new names fresh for the variables occurring in the rules.

Here we need closedness. [FG10]

Closedness

Intuitively, no free atom occurs in a closed term – closed axioms do not allow abstracted atoms to become free.

If t is a term, we say that $t^{\text{И}}$ is a freshened variant of t when $t^{\text{И}}$ has the same structure as t, except that the atoms and unknowns have been replaced by 'fresh' atoms and unknowns.

$$[a][b]X : \qquad [a^{\text{И}}][b^{\text{И}}]X^{\text{И}} \qquad [a^{\text{И}}][a^{\text{И}}]X^{\text{И}} \qquad [a^{\text{И}}][b^{\text{И}}]X$$

Closedness

Intuitively, no free atom occurs in a closed term – closed axioms do not allow abstracted atoms to become free.

If t is a term, we say that $t^{\text{И}}$ is a freshened variant of t when $t^{\text{И}}$ has the same structure as t, except that the atoms and unknowns have been replaced by 'fresh' atoms and unknowns.

$$[a][b]X : \qquad [a^{\text{И}}][b^{\text{И}}]X^{\text{И}} \qquad [a^{\text{И}}][a^{\text{И}}]X^{\text{И}} \qquad [a^{\text{И}}][b^{\text{И}}]X$$

Closed term (in-context)

A term-in-context $\nabla \vdash l$ is closed if there exists a solution for the matching problem

$$(\nabla^{\text{И}} \vdash l^{\text{И}}) \;{}_?{\approx}\; (\nabla, A(\nabla^{\text{И}}, l^{\text{И}})\#V(\nabla, l) \vdash l).$$

Extending results

Closed Nominal Rewriting modulo C

The *one-step closed rewrite modulo* C *relation* $\Delta \vdash s \overset{\mathsf{R,C}}{\longrightarrow}_c t$ is the least relation such that for any $R = (\nabla \vdash l \to r) \in \mathsf{R}$ and term-in-context $\Delta \vdash s$, there is some $R^{\text{\reflectbox{N}}}$ a freshened variant of R (so fresh for R, Δ, s, t), position $\mathtt{C}$, term s', permutation π, and substitution θ,

$$\frac{s \equiv \mathtt{C}[s'] \qquad \Delta, A(R^{\text{\reflectbox{N}}})\#V(\Delta, s, t) \vdash (\nabla^{\text{\reflectbox{N}}}\theta,\ s' \approx_{\alpha,\mathsf{C}} l^{\text{\reflectbox{N}}}\theta,\ \mathtt{C}[r^{\text{\reflectbox{N}}}\theta] \approx_{\alpha,\mathsf{C}} t)}{\Delta \vdash s \to^{c}_{R,\mathsf{C}} t}$$

Problem fixed

$$\begin{tikzcd}[column sep=small]
& \Delta' \vdash l_3\theta = P_3 \wedge \exists[a_3](\forall[b_7]\pi \cdot Q_7) \arrow[dl, "a_3\#P_3"', "R_3", "c" near end] \arrow[dr, "R_7"', "c"' near end] & \\
\Delta' \vdash \exists[a_3](P_3 \wedge \forall[b_7]\pi \cdot Q_7) & & \Delta' \vdash P_3 \wedge \forall[b_7](\exists[a_3]\pi \cdot Q_7)
\end{tikzcd}$$

Problem fixed

$$
\begin{tikzcd}[column sep=small]
& \Delta' \vdash l_3\theta = P_3 \land \exists[a_3](\forall[b_7]\pi \cdot Q_7) \arrow[dl, "a_3\#P_3"', "R_3", "c" near end] \arrow[dr, "R_7"', "c" near end] & \\
\Delta' \vdash \exists[a_3](P_3 \land \forall[b_7]\pi \cdot Q_7) \arrow[d, "a^{\text{и}}_1\#P_3"', "R_1", "c" near end] & & \Delta' \vdash P_3 \land \forall[b_7](\exists[a_3]\pi \cdot Q_7) \arrow[d, "R_1"', "a^{\text{и}}_1\#P_3", "c" near end] \\
\Delta' \vdash \exists[a_3](\forall[a^{\text{и}}_1](P_3 \land (a^{\text{и}}_1\ b_7)\pi \cdot Q_7)) & & \Delta' \vdash \forall[a^{\text{и}}_1](P_3 \land \exists[a_3](a^{\text{и}}_1\ b_7)\pi \cdot Q_7)
\end{tikzcd}
$$

Problem fixed

$$
\begin{tikzcd}[column sep=small]
 & \Delta' \vdash l_3\theta = P_3 \wedge \exists[a_3](\forall[b_7]\pi \cdot Q_7) \arrow[dl, "R_3", "a_3\#P_3"', "c" near end] \arrow[dr, "R_7"', "c"' near end] & \\
\Delta' \vdash \exists[a_3](P_3 \wedge \forall[b_7]\pi \cdot Q_7) \arrow[d, "R_1", "a_1^{\text{И}}\#P_3"', "c" near end] & & \Delta' \vdash P_3 \wedge \forall[b_7](\exists[a_3]\pi \cdot Q_7) \arrow[d, "R_1"', "a_1^{\text{И}}\#P_3", "c" near end] \\
\Delta' \vdash \exists[a_3](\forall[a_1^{\text{И}}](P_3 \wedge (a_1^{\text{И}}\ b_7)\pi \cdot Q_7)) \arrow[dr, "R_7", "c"' near end] & & \Delta' \vdash \forall[a_1^{\text{И}}](P_3 \wedge \exists[a_3](a_1^{\text{И}}\ b_7)\pi \cdot Q_7) \arrow[dl, "R_3"', "a_3\#P_3", "c" near end] \\
 & \Delta' \vdash \forall[a_1^{\text{И}}](\exists[a_3](P_3 \wedge (a_1^{\text{И}}\ b_7)\pi \cdot Q_7)) &
\end{tikzcd}
$$

Conclusion and Future Work

- Closedness is essential to guarantee the confluence of this particular NRS – it simplifies the computation of critical pairs.
- A nominal critical pair modulo C is a new concept that is under investigation:
  - we still need to prove a version of the nominal Critical Pair Lemma modulo C.
- We want to apply the current extensions in the development of closed nominal narrowing modulo C:
  - we have to prove a version of the nominal Lifting Theorem modulo C.

References

References

Mauricio Ayala-Rincón, Washington de Carvalho Segundo, Maribel Fernández, and Daniele Nantes-Sobrinho.
On solving nominal fixpoint equations.
In Clare Dixon and Marcelo Finger, editors, *Frontiers of Combining Systems - 11th International Symposium, FroCoS 2017, Brasília, Brazil, September 27-29, 2017, Proceedings*, volume 10483 of *Lecture Notes in Computer Science*, pages 209–226. Springer, 2017.

Mauricio Ayala-Rincón, Maribel Fernández, and Daniele Nantes-Sobrinho.
Nominal narrowing.
In Delia Kesner and Brigitte Pientka, editors, *1st International Conference on Formal Structures for Computation and Deduction, FSCD 2016, June 22-26, 2016, Porto, Portugal*, volume 52 of *LIPIcs*, pages 11:1–11:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

Maribel Fernández and Murdoch James Gabbay.
Closed nominal rewriting and efficiently computable nominal algebra equality.
In Karl Crary and Marino Miculan, editors, *Proceedings 5th International Workshop on Logical Frameworks and Meta-languages: Theory and Practice, LFMTP 2010, Edinburgh, UK, 14th July 2010*, volume 34 of *EPTCS*, pages 37–51, 2010.

THE
END

Appendix

Simplification rules for C-unification

We follow the approach by Ayala et. al. [AdCSFN17].

(#ab)	$(\Delta,\theta,Pr\uplus\{a\#b\})$	$\Longrightarrow$	(Δ,θ,Pr)
(#app)	$(\Delta,\theta,Pr\uplus\{a\#f(t_1,\cdots,t_n)\})$	$\Longrightarrow$	$(\Delta,\theta,Pr\cup\{a\#t_1,\cdots,a\#t_n\})$
(#a[a])	$(\Delta,\theta,Pr\uplus\{a\#[a]t\})$	$\Longrightarrow$	(Δ,θ,Pr)
(#a[b])	$(\Delta,\theta,Pr\uplus\{a\#[b]t\})$	$\Longrightarrow$	$(\Delta,\theta,Pr\cup\{a\#t\})$
(#var)	$(\Delta,\theta,Pr\uplus\{a\#\pi\cdot X\})$	$\Longrightarrow$	$(\{(\pi^{-1}\cdot a)\#X\}\cup\Delta,\theta,Pr)$

$(\approx_{\alpha,\mathrm{C}}$ refl)	$(\Delta,\theta,Pr\uplus\{s\approx_{\alpha,\mathrm{C}} s\})$	$\Longrightarrow$	(Δ,θ,Pr)
$(\approx_{\alpha,\mathrm{C}}$ app)	$(\Delta,\theta,Pr\uplus\{f(\overline{s})_n\approx_{\alpha,\mathrm{C}} f(\overline{t})_n\})$	$\Longrightarrow$	$(\Delta,\theta,Pr\cup\bigcup\{s_i\approx_{\alpha,\mathrm{C}} t_i\})$
$(\approx_{\alpha,\mathrm{C}}\ C)$	$(\Delta,\theta,Pr\uplus\{f^{\mathrm{C}}s\approx_{\alpha,\mathrm{C}} f^{\mathrm{C}}t\})$	$\Longrightarrow$	$(\Delta,\theta,Pr\cup\{s\approx_{\alpha,\mathrm{C}} v\})$, where $s=(s_0,s_1)$ and $t=(t_0,t_1)$, $v=(t_i,t_{(i+1)mod 2})$, $i=0,1$
$(\approx_{\alpha,\mathrm{C}}$ [aa])	$(\Delta,\theta,Pr\uplus\{[a]s\approx_{\alpha,\mathrm{C}} [a]t\})$	$\Longrightarrow$	$(\Delta,\theta,Pr\cup\{s\approx_{\alpha,\mathrm{C}} t\})$
$(\approx_{\alpha,\mathrm{C}}$ [ab])	$(\Delta,\theta,Pr\uplus\{[a]s\approx_{\alpha,\mathrm{C}} [b]t\})$	$\Longrightarrow$	$(\Delta,\theta,Pr\cup\{s\approx_{\alpha,\mathrm{C}} (a\ b)\cdot t, a\#t\})$
$(\approx_{\alpha,\mathrm{C}}$ inst)	$(\Delta,\theta,Pr\uplus\{\pi\cdot X\approx_{\alpha,\mathrm{C}} t\})$	$\Longrightarrow$	$(\Delta,\theta',Pr[X\mapsto\pi^{-1}\cdot t]\cup\bigcup_{\substack{Y\in dom(\theta'),\\ a\#Y\in\Delta}}\{a\#Y\theta'\})$, let $\theta':=\theta[X\mapsto\pi^{-1}\cdot t]$, if $X\notin Var(t)$
$(\approx_{\alpha,\mathrm{C}}$ inv)	$(\Delta,\theta,Pr\uplus\{\pi\cdot X\approx_{\alpha,\mathrm{C}} \pi'\cdot X\})$	$\Longrightarrow$	$(\Delta,\theta,Pr\cup\{\pi\oplus(\pi')^{-1}\cdot X\approx_{\alpha,\mathrm{C}} X\})$ if $\pi'\neq\mathtt{Id}$

Nominal Rewriting

Nominal rewriting

The *one-step rewrite relation* $\Delta \vdash s \xrightarrow{\mathsf{R}}_{[\mathsf{C},R,\theta,\pi]} t$ is the least relation such that for any $R = (\nabla \vdash l \to r) \in \mathsf{R}$, position C, term s', permutation π, and substitution θ,

$$\frac{s \equiv \mathsf{C}[s'] \qquad \Delta \vdash \left(\nabla\theta, s' \approx_\alpha \pi \cdot (l\theta), \mathsf{C}[\pi \cdot (r\theta)] \approx_\alpha t\right)}{\Delta \vdash s \xrightarrow{\mathsf{R}}_{[\mathsf{C},R,\theta,\pi]} t}$$

- To find θ and π above, we need to solve the nominal matching problem $(\Delta \vdash s') \approx_? (\nabla \vdash l)$.

- A NRS is said to be *confluent* when for all Δ, s, t and t' such that $\Delta \vdash s \rightarrow^* t$ and $\Delta \vdash s \rightarrow^* t'$, there exists u such that $\Delta \vdash t \rightarrow^* u$ and $\Delta \vdash t' \rightarrow^* u$.

Notice we need the same Δ here. We will find some complications later.

Since atoms are not affected by substitution actions but can be swapped, we need to consider a technicality called *equivariance*.

- The *equivariant closure* of a set *Rw* of rewrite rules is the closure of *Rw* by the meta-action of permutations, that is, it is the set of all permutative variants of rules in *Rw*. We denote *eq-closure(Rw)* for the equivariant closure of *Rw*.

Consider the NRS with the single rule $R \equiv a\#X \vdash f(X,b) \longrightarrow a$. In order to find the *eq-closure(Rw)*, we need to analyze all the permutative variants of $R \in Rw$, they are $R^{(a\ b)}$, $R^{(a\ c)}$, $R^{(b\ c)}$ and $R^{(a\ c)(b\ d)}$, where c, d are arbitrary new atoms.

$$R_1 = R^{(a\ b)} = b\#X \vdash f(X,a) \longrightarrow b$$

$$R_2 = R^{(a\ c)} = c\#X \vdash f(X,b) \longrightarrow c$$

$$R_3 = R^{(b\ c)} = a\#X \vdash f(X,c) \longrightarrow a$$

$$R_4 = R^{(a\ c)(b\ d)} = c\#X \vdash f(X,d) \longrightarrow c$$

Therefore, *eq-closure(Rw)* $= \{R, R_1, R_2, R_3, R_4\}$.

Critical Pairs

(Permutative overlaps and critical pairs)

Let $R_1 = \nabla_1 \vdash l_1 \longrightarrow r_1$ and $R_2 = \nabla_2 \vdash l_2 \longrightarrow r_2$ be copies of two rewrite rules in *eq-closure(Rw)* such that there is an overlap.

If R_2 is a copy of R_1^π, we say that the overlap is *permutative*.

A permutative overlap at the root position is called *root-permutative*.

We call an overlap that is not trivial and not root-permutative *proper*.

The same terminology is used to classify critical pairs.

Critical Pairs

(Peak and local confluence)

Let R be an equivariant rewrite system, and let Δ, s, t_1 and t_2 such that $\Delta \vdash s \rightarrow t_1$ and $\Delta \vdash s \rightarrow t_2$. This pair will be denoted as $\Delta \vdash s \rightarrow t_1, t_2$ and called a *peak*.

If there is such a peak, then we call a NRS *locally confluent* when there exists a term u such that $\Delta \vdash t_1 \rightarrow^* u$ and $\Delta \vdash t_2 \rightarrow^* u$. We say such a peak is *joinable*.

Notice we need the same Δ here again.

In this way, we can only say that a critical pair is joinable if its terms are under the same context.

Main Problem

Let $\Delta = \{a_3 \# P_3\}$.

$$
\begin{aligned}
&(\Delta, \emptyset, \{l_3|_2 \;{}_?{\approx}_?\; l_7\}) = \\
&= (\Delta, \emptyset, \{\exists[a_3]Q_3 \;{}_?{\approx}_?\; \exists[a_7](\forall[b_7]Q_7)\}) \\
&\Rightarrow_{(\approx_{\alpha,\mathsf{C}}\mathsf{app})} (\Delta, \emptyset, \{[a_3]Q_3 \;{}_?{\approx}_?\; [a_7](\forall[b_7]Q_7)\}) \\
&\Rightarrow_{(\approx_{\alpha,\mathsf{C}}[ab])} (\Delta, \emptyset, \{Q_3 \;{}_?{\approx}_?\; (a_3\ a_7) \cdot \forall[b_7]Q_7, a_3 \# \forall[b_7]Q_7\}) \\
&\Rightarrow^2_{(\#\mathsf{app})} (\Delta, \emptyset, \{Q_3 \;{}_?{\approx}_?\; \forall[b_7](a_3\ a_7) \cdot Q_7, a_3 \# Q_7\}) \\
&\Rightarrow_{(\approx_{\alpha,\mathsf{C}}\mathtt{inst})} (\Delta, \theta = [Q_3 \mapsto \forall[b_7](a_3\ a_7) \cdot Q_7], \\
&\qquad\qquad \{\forall[b_7](a_3\ a_7) \cdot Q_7 \;{}_?{\approx}_?\; \forall[b_7](a_3\ a_7) \cdot Q_7, a_3 \# Q_7\}) \\
&\Rightarrow_{(\approx_{\alpha,\mathsf{C}}\mathtt{refl})} (\Delta, \theta, \{a_3 \# Q_7\}) \\
&\Rightarrow_{(\#\mathsf{var})} (\Delta \cup \{a_3 \# Q_7\}, \theta, \emptyset)
\end{aligned}
$$

Nominal rewriting not complete for equational reasoning

Suppose R is a presentation of E. It is **not** necessarily the case that

$$\Delta \vdash_{\mathsf{E}} s = t \quad \text{implies} \quad \Delta \vdash_{\mathsf{R}} s \leftrightarrow t.$$

Take $\mathsf{E} = \{a\#X \vdash X = f(X)\}$ and $\mathsf{R} = \{a\#X \vdash X \rightarrow f(X)\}$.

Then we have $\vdash_{\mathsf{E}} X = f(X)$ by definition, using $\Gamma = a\#X$, but $\nvdash_{\mathsf{R}} X \leftrightarrow f(X)$.

Nominal Narrowing [AFN16]

Nominal Narrowing

The *one-step narrowing relation* $(\Delta \vdash s) \rightsquigarrow_{[\mathtt{C},R,\theta,\pi]} (\Delta' \vdash t)$ is the least relation such that for any $R = (\nabla \vdash l \to r) \in \mathsf{R}$, position $\mathtt{C}$, term s', permutation π, and substitution θ,

$$\frac{s \equiv \mathtt{C}[s'] \qquad \Delta' \vdash \left(\nabla\theta,\ \Delta\theta,\ s'\theta \approx_\alpha \pi \cdot (l\theta),\ (\mathtt{C}[\pi \cdot r])\theta \approx_\alpha t\right)}{(\Delta \vdash s) \rightsquigarrow_{[\mathtt{C},R,\theta,\pi]} (\Delta' \vdash t)}$$

- To find θ and π above, we need to solve the nominal unification problem $(\Delta \vdash s') \;{}_?{\approx}_?\; (\nabla \vdash l)$.

Definition closedness

Closed rewriting

The *one-step closed rewrite relation* $\Delta \vdash s \overset{\mathsf{R}}{\longrightarrow}_c t$ is the least relation such that for any $R = (\nabla \vdash l \longrightarrow r) \in \mathsf{R}$ and term-in-context $\Delta \vdash s$, there is some $R^{\text{И}}$ a freshened variant of R (so fresh for R, Δ, s, t), position $\mathtt{C}$, term s', permutation π, and substitution θ,

$$\frac{s \equiv \mathtt{C}[s'] \qquad \Delta, A(R^{\text{И}})\#V(\Delta, s, t) \vdash (\nabla^{\text{И}}\theta,\ s' \approx_\alpha l^{\text{И}}\theta,\ C[r^{\text{И}}\theta] \approx_\alpha t)}{\Delta \vdash s \longrightarrow_R^c t}$$