Mechanising Combinatorial Applications of Compactness

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Talk's Plan

Contextualisation

Compactness Theorem[Gödel (Satz X) 1930] A set of first-order sentences has a model if and only if every finite subset of it has a model.

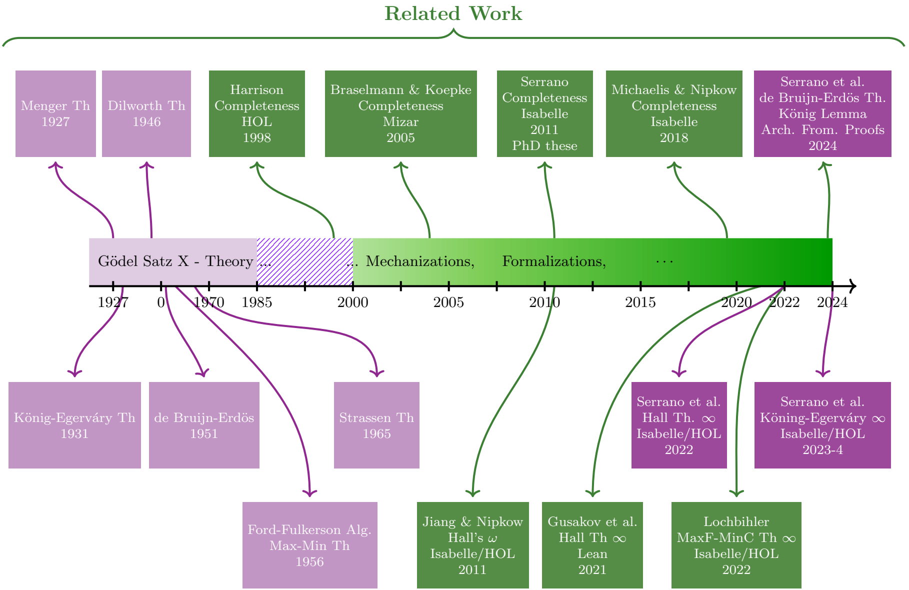
Typical textbook proofs infer the compactness theorem from Gödel's completeness theorem!

Paseau and Leek (pag. 10 in The Compactness Theorem):

> *"proofs of compactness via completeness are not satisfactory because they are based on properties incidental to the semantic property of interest. Such proofs conclude compactness, a semantic property, from a property of the logic relating its syntax to its semantics."*

"From the perspective of a model theorist who sees talk of syntax as a heuristic for the study of certain relations between structures that happen to have syntactic correlates, proving compactness via completeness is tantamount to heresy (page 53 in Poizat's textbook A Course in Model Theory)."

Contextualisation



Hall's Theorem - finite case

- Proved by Philip Hall in 1935: a condition that guarantees the existence of a System of Distinct Representatives (SDR) for a finite collection of finite sets:
  - Given a finite collection of finite sets $\{S_i\}_{i\in I}$.
  - An SDR is a sequence of distinct elements $(x_i)_{i\in I}$, such $x_i \in S_i$.

Theorem (Hall's Theorem — finite case)

Consider a finite collection $\{S_1, S_2, \ldots, S_n\}$ of finite subsets of an arbitrary set S.

The collection $\{S_1, S_2, \ldots, S_n\}$ has an SDR

if and only if

for every $1 \leq k \leq n$ and an arbitrary set of k distinct indices $1 \leq i_1, \ldots, i_k \leq n$, one has that $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$. *(M)*

(M) is the so-called *marriage condition*.

Hall's Theorem - finite case: Formalisation in Isabelle/HOL

Jiang and Nipkow (Certified Programs and Proofs 2011) formalised the finite case of Hall's theorem in Isabelle/HOL.

$\Rightarrow$ Mechanisations of the proofs by

- Halmos and Vaughan's (Am. J. Math 1950) and
- Rado (Lond. Math. Soc. 1967).

Hall's Theorem - countable (infinite) case

Theorem (Hall's Theorem — countable case)

Let $\{S_i\}_{i \in I}$ be a collection of finite subsets of an arbitrary set S, where I is a countable set of indices.

The collection $\{S_i\}_{i \in I}$ has an SDR

if and only if

For every finite subset of indices $J \subseteq I$, $|\bigcup_{j \in J} S_j| \geq |J|$. (M^*)

⚙ Serrano, de Lima and Ayala-Rincón formalised this result (Congress on Intelligent Computer Mathematics 2022)📘

$\Rightarrow$ The mechanisation applies the formalisation of the Compactness Theorem for propositional logic in Serrano's PhD thesis (2011) 📘 and verifies the marriage condition for finite families using Jiang and Nipkow's formalisation.

Hall Theorem - Formalisation Approach

Consider the propositional language with constant symbols given by the set below

$$\mathcal{P} = \{C_{i,x} \mid i \in I, x \in S_i\}$$

$C_{i,x}$ is interpreted as "select the element x from the set S_i."
The sets of propositional formulas describe the existence of an SDR for $\{S_i\}_{i \in I}$:

1. "Select at least an element from each S_i:"

$$\mathcal{F} = \{\vee_{x \in S_i} C_{i,x} \mid i \in I\}.$$

2. "Select at most an element from each S_i:"

$$\mathcal{G} = \{\neg(C_{i,x} \wedge C_{i,y}) \mid x, y \in S_i, x \neq y, i \in I\}.$$

3. "Do not select more than once the same element from $\bigcup_{i \in I} S_i$:"

$$\mathcal{H} = \{\neg(C_{i,x} \wedge C_{j,x}) \mid x \in S_i \cap S_j, i \neq j, i, j \in I\}.$$

Assuming the marriage condition (M^*), the Compactness Theorem is used to prove satisfiability of

$$\mathcal{T} = \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$$

Hall Theorem - Formalisation Approach

Let $\mathcal{T}_0$ be any finite subset of formulas in $\mathcal{T}$ and let $J = \{i_1, \ldots, i_m\} \subset I$ the finite subset of indices "referred" in $\mathcal{T}_0$.
$\mathcal{T}_0$ is contained in the set

$$\mathcal{T}_1 = \mathcal{F}_0 \cup \mathcal{G}_0 \cup \mathcal{H}_0, \text{ where}$$

1. $\mathcal{F}_0 = \left\{ \bigvee_{x \in S_j} C_{j,x} \mid j \in J \right\}$,
2. $\mathcal{G}_0 = \{\neg(C_{j,x} \wedge C_{j,y}) \mid x, y \in S_j, x \neq y, j \in J\}$,
3. $\mathcal{H}_0 = \{\neg(C_{j,x} \wedge C_{k,x}) \mid x \in S_j \cap S_k, j \neq k, j, k \in J\}$.

By hypothesis, $\{S_{i_1}, \ldots, S_{i_m}\}$ satisfies the marriage condition (M), and by the finite version of Hall's Theorem there exists a function $f : J \to \bigcup_{i \in J} S_i$ such that the image of f gives an SDR for $\{S_{i_1}, \ldots, S_{i_m}\}$.

Hall Theorem - Formalisation Approach

A model for $\mathcal{T}_1$ is given by the interpretation $v : \mathcal{P} \to \{\mathsf{V}, \mathsf{F}\}$ defined by,

$$v(C_{j,x}) = \begin{cases} \mathsf{V}, & \text{if } j \in J \text{ and } f(j) = x, \\ \mathsf{F}, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{T}_1$ is satisfiable and so is $\mathcal{T}_0$.

Therefore, by the Compactness Theorem, $\mathcal{T}$ is satisfiable.

Formalisation

Definition *system-representatives* :: $('a \Rightarrow 'b\ set) \Rightarrow 'a\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow$ *bool* **where**
$\textit{system-representatives}\ S\ I\ R \ \equiv (\forall\, i{\in}I.\ (R\ i) \in (S\ i)) \wedge (\textit{inj-on}\ R\ I)$

Above, *(inj-on R I)* means that the function R is injective on I.

The marriage condition for S and I is formalized by the proposition,

$$\forall J \subseteq I.\, \textit{finite}\, J \longrightarrow \textit{card}\, J \leq \textit{card}\left(\bigcup (S\ `\ J)\right)$$

where $S\ `\ J = \{S\,j \,|\, j \in J\}$.

Formalisation

Definition $\mathcal{F}$ :: $('a \Rightarrow 'b\ set) \Rightarrow 'a\ set \Rightarrow (('a \times 'b) formula)\ set$ **where**

$$\mathcal{F}\ S\ I \equiv (\bigcup i{\in}I.\ \{\ disjunction\text{-}atomic\ (set\text{-}to\text{-}list\ (S\ i))\ i\ \})$$

Definition $\mathcal{G}$ :: $('a \Rightarrow 'b\ set) \Rightarrow 'a\ set \Rightarrow ('a \times 'b) formula\ set$ **where**

$$\mathcal{G}\ S\ I \equiv \{\neg.(atom\ (i,x) \wedge.\ atom(i,y))\ |x\ y\ i\ .\ x{\in}(S\ i) \wedge y{\in}(S\ i) \wedge\ x{\neq}y \wedge i{\in}I\}$$

Definition $\mathcal{H}$ :: $('a \Rightarrow 'b\ set) \Rightarrow 'a\ set \Rightarrow ('a \times 'b) formula\ set$ **where**

$$\mathcal{H}\ S\ I \equiv \{\neg.(atom\ (i,x) \wedge.\ atom(j,x))\ |\ x\ i\ j.\ x \in (S\ i) \cap (S\ j) \wedge (i{\in}I \wedge j{\in}I \wedge i{\neq}j)\}$$

Formalisation

Lemma *system-distinct-representatives-finite*:
assumes
$\forall\, i{\in}I.\ (S\ i){\neq}\{\}$ **and** $\forall\, i{\in}I.\ \mathit{finite}\ (S\ i)$ **and** $\mathit{To} \subseteq (\mathcal{T}\ S\ I)$ **and** $\mathit{finite}\ \mathit{To}$
and $\forall\, J{\subseteq}(\mathit{indices\text{-}set\text{-}formulas}\ \mathit{To}).\ \mathit{card}\ J \leq \mathit{card}\ (\bigcup\ (S\ \text{`}\ J))$
shows $\exists\, R.\ \mathit{system\text{-}representatives}\ S\ (\mathit{indices\text{-}set\text{-}formulas}\ \mathit{To})\ R$

Lemma *SDR-satisfiable*:
assumes $\forall\, i{\in}I.\ (A\ i) \neq \{\}$ **and** $\forall\, i{\in}I.\ \mathit{finite}\ (A\ i)$ **and** $X \subseteq (\mathcal{T}\ A\ I)$
and $\mathit{system\text{-}representatives}\ A\ I\ R$
shows $\mathit{satisfiable}\ X$

Lemma *finite-is-satisfiable*:
assumes
$\forall\, i{\in}I.\ (S\ i){\neq}\{\}$ **and** $\forall\, i{\in}I.\ \mathit{finite}\ (S\ i)$ **and** $\mathit{To} \subseteq (\mathcal{T}\ S\ I)$ **and** $\mathit{finite}\ \mathit{To}$
and $\forall\, J{\subseteq}(\mathit{indices\text{-}set\text{-}formulas}\ \mathit{To}).\ \mathit{card}\ J \leq \mathit{card}\ (\bigcup\ (S\ \text{`}\ J))$
shows $\mathit{satisfiable}\ \mathit{To}$

Formalisation

Lemma *all-formulas-satisfiable*:
fixes $S :: 'a \Rightarrow 'b\ set$ **and** $I :: 'a\ set$
assumes $\exists g.\ enumeration\ (g{::}\ nat \Rightarrow 'a)$ **and** $\exists h.\ enumeration\ (h{::}\ nat \Rightarrow 'b)$
and $\forall i{\in}I.\ finite\ (S\ i)$
and $\forall J{\subseteq}I.\ finite\ J \longrightarrow\ card\ J \leq card\ (\bigcup\ (S\ `\ J))$
shows $satisfiable\ (\mathcal{T}\ S\ I)$

Lemma *function-SDR*:
assumes $i \in I$ **and** $M\ model\ (\mathcal{F}\ S\ I)$ **and** $M\ model\ (\mathcal{G}\ S\ I)$ **and** $finite(S\ i)$
shows $\exists!x.\ (value\ M\ (atom\ (i,x)) = Ttrue) \wedge\ x \in (S\ i) \wedge SDR\ M\ S\ I\ i = x$

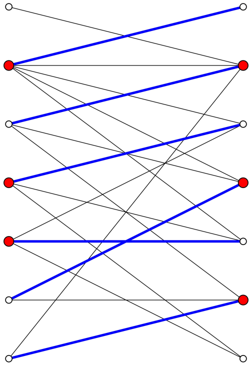
Theorem *Hall*:
fixes $S :: 'a \Rightarrow 'b\ set$ **and** $I :: 'a\ set$
assumes $\exists g.\ enumeration\ (g{::}\ nat \Rightarrow 'a)$ **and** $\exists h.\ enumeration\ (h{::}\ nat \Rightarrow 'b)$
and *Finite*: $\forall i{\in}I.\ finite\ (S\ i)$
and *Marriage*: $\forall J{\subseteq}I.\ finite\ J \longrightarrow\ card\ J \leq card\ (\bigcup\ (S\ `\ J))$
shows $\exists R.\ system\text{-}representatives\ S\ I\ R$
proof−

König-Egerváry theorem

Formalisations deriving from Hall Theorem:

- König-Egerváry theorem

"In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover."



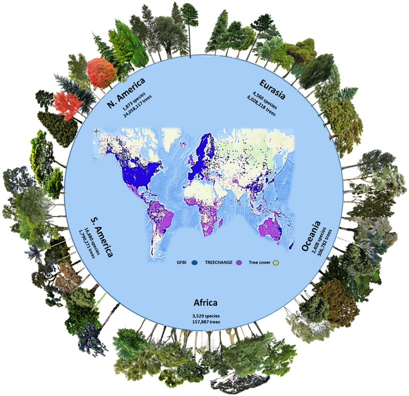
Taken from Wikipedia, by David Eppstein

A matching that covers all left-vertices gives an SDR for the infinite collection of sets given by right-vertices incident to each left-vertex.

Hall's Theorem: Graph-Theoretical version - König-Egerváry theorem

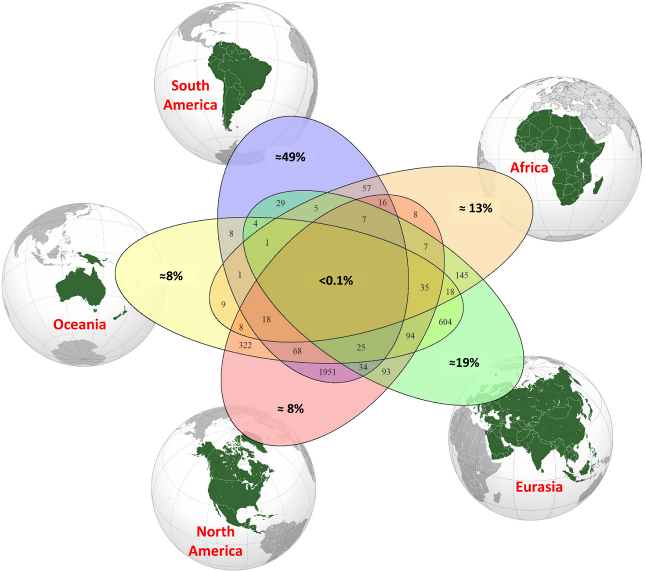
Example Let $\{T_c\}_{c \in C}$ be the collection of sets of tree species inhabiting each country, from the estimated 73.000 tree species in the world.

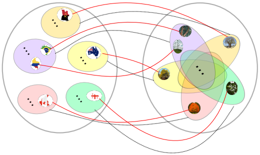
Select a different national tree for each country in the world: $(t_c)_{c \in C}$



The number of tree species on Earth PNAS 119(6), 2022

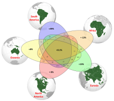
König-Egerváry Theorem

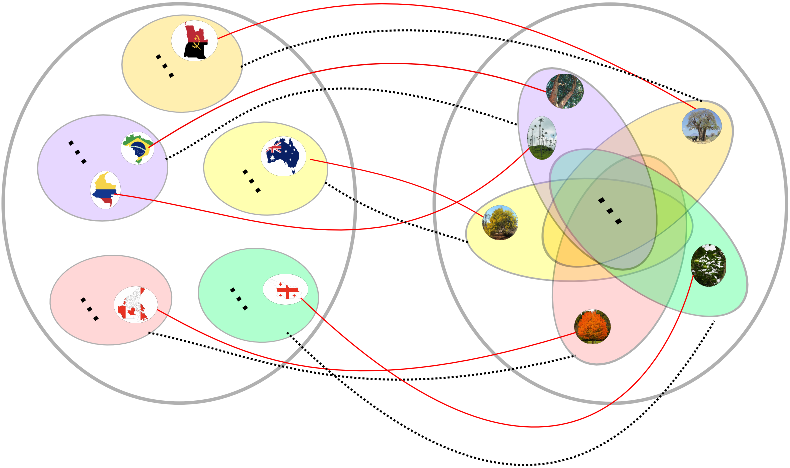




The number of tree species on Earth PNAS 119(6), 2022 https://doi.org/10.1016/S1385-7258(51)50053-7

König-Egerváry Theorem





Perfect match on the bipartite graph $G = \langle C, \bigcup_{c \in C} T_c, E \rangle$ covering C

König-Egerváry Theorem countable (infinite) version

Theorem (Hall's Theorem - graph-theoretical countable version)

Let $G = \langle X, Y, E\rangle$ be a countable bipartite graph, where for all $x \in X$, $N(x)$ is finite.

A perfect matching covering X exists

if and only if

for all J finite, $J \subseteq X$, $|J| \leq |N(J)|$. $\quad (M_G)$

Formalisation Approach: From sets to graphs

SDR associated to a directed bipartite digraph

Let $G = \langle X, Y, E \rangle$ be a directed bipartite digraph.
The collection of sets associated with G is built as

$$\{V_i\}_{i \in X},$$

where for all $i \in X$.

$$V_i = \{y \mid (i, y) \in E\}$$

Therefore, if $E' \subseteq E$ is a perfect matching covering X, the function

$$R : X \to \bigcup_{i \in X} V_i, \text{ defined as } R(i) \mapsto y \text{ s.t. } (i, y) \in E'$$

is an SDR of the set collection $\{V_i\}_{i \in X}$.

Formalisation Approach: From sets to graphs

```
definition bipartite_digraph:: "('a,'b) pre_digraph ⇒ 'a set ⇒ 'a set ⇒ bool"
   "bipartite_digraph G X Y  ≡
       (X ∪ Y = (verts G)) ∧  X ∩ Y = {} ∧
       (∀e ∈ (arcs G).(tail G e) ∈ X ⟷ (head G e) ∈ Y)"

 (* Matchings in directed bipartite digraphs *)
 definition dirBD_matching:: "('a,'b) pre_digraph ⇒ 'a set ⇒ 'a set ⇒ 'b set ⇒ bool"
   where
   "dirBD_matching G X Y E  ≡
           dir_bipartite_digraph G X Y ∧ (E  ⊆  (arcs G)) ∧
           (∀ e1∈E. (∀ e2∈ E.  e1 ≠ e2 ⟶
           ((head G e1) ≠ (head G e2)) ∧
           ((tail G e1) ≠ (tail G e2))))"

(* Perfect matching (covering tail vertexes) in directed bipartite digraphs *)
definition dirBD_perfect_matching::
  "('a,'b) pre_digraph ⇒ 'a set ⇒ 'a set ⇒ 'b set ⇒ bool"
  where
  "dirBD_perfect_matching G X Y E ≡
   dirBD_matching G X Y E ∧ (tails_set G E = X)"
```

Formalisation Approach: From sets to graphs

```
definition E_head :: "('a,'b) pre_digraph ⇒ 'b set  ⇒ ('a  ⇒ 'a)"
  where
  "E_head G E = (λx. (THE y. ∃ e. e ∈ E ∧ tail G e = x ∧  head G e = y))"

definition dirBD_to_Hall::
    "('a,'b) pre_digraph ⇒ 'a set ⇒ 'a set ⇒ 'a set ⇒ ('a  ⇒ 'a set) ⇒ bool"
    where
    "dirBD_to_Hall G X Y I S ≡
    dir_bipartite_digraph G X Y ∧ I = X ∧ (∀v∈I. (S v) = (neighbourhood G v))"

theorem dir_BD_to_Hall:
    "dirBD_perfect_matching G X Y E ⟶
     system_representatives (neighbourhood G) X (E_head G E)"
```

Formalisation Approach: From graphs to sets

Perfect matching associated to a collection of sets

Let $\{S_i\}_{i\in I}$ be a collection of subsets of an arbitrary set S.
The associated directed bipartite digraph is built as the graph

$$G = \langle I, Y, E\rangle,$$

where $Y = \bigcup_{i\in I} S_i$ and $E = \{(i, y) \mid i \in I \text{ and } y \in S_i\}$.

Therefore, if R is an SDR of $\{S_i\}_{i\in I}$, the subset of arcs

$$E' = \{(x, y) \mid x \in I \text{ and } y = R(x)\}$$

is a perfect matching covering I.

Formalisation Approach: From graphs to sets

```
lemma marriage_necessary_graph:
  assumes "(dirBD_perfect_matching G X Y E)" and "∀i∈X. finite (neighbourhood G i)"
  shows "∀J⊆X. finite J ⟶ (card J) ≤ card (⋃ (neighbourhood G ` J))"
```

```
lemma marriage_sufficiency_graph:
  fixes  G :: "('a::countable, 'b::countable) pre_digraph" and X:: "'a set"
  assumes "dir_bipartite_digraph G X Y"  and "∀i∈X. finite (neighbourhood G i)"
  shows
  "(∀J⊆X. finite J ⟶ (card J) ≤ card (⋃ (neighbourhood G ` J))) ⟶
   (∃E. dirBD_perfect_matching G X Y E)"
```

Formalisation Approach: From graphs to sets

```
(* Graph version of Hall's Theorem *)

theorem Hall_digraph:
 fixes  G :: "('a::countable, 'b::countable) pre_digraph" and X:: "'a set"
  assumes "dir_bipartite_digraph G X Y"  and "∀i∈X. finite (neighbourhood G i)"
  shows "(∃E. dirBD_perfect_matching G X Y E) ⟷
  (∀J⊆X. finite J ⟶ (card J) ≤ card (⋃ (neighbourhood G ` J)))"
```

Additional results and work in Progress

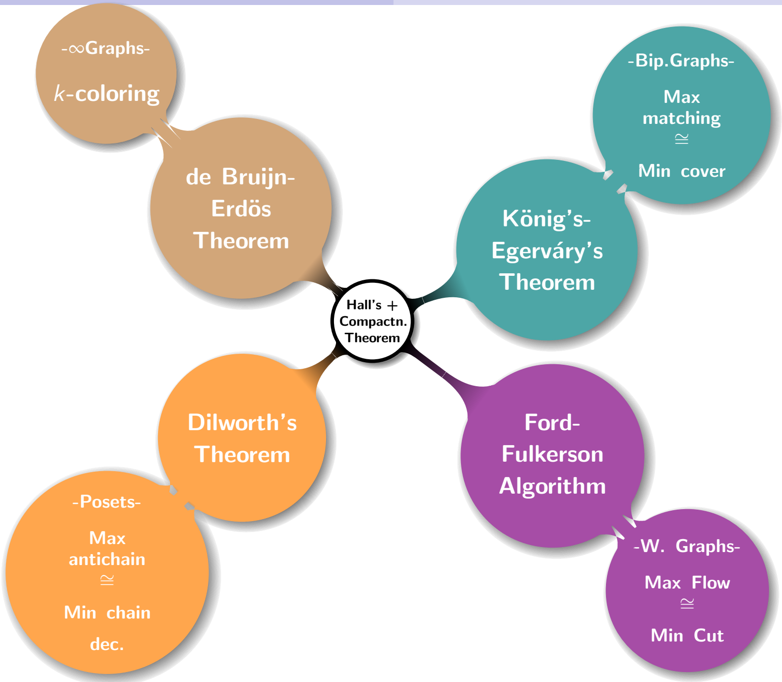
Other formalisations available in the Isabelle distribution and based on the Compactness Theorem:

- De Bruijn-Erdös's graph colouring theorem Ind. Math (1951)

> "The chromatic number of a graph equals n if and only if the chromatic numbers of all its finite subgraphs are $\leq n$."

- König's lemma (cf exercise in Chapter I.6 in Nerode and Shore's *Logic for Applications* textbook (2012).

> "A finitely branching tree is infinite iff it has an infinite path."



-∞Graphs-
k-coloring
de Bruijn-Erdös Theorem
-Bip.Graphs-
Max matching
≅
Min cover
König's-Egerváry's Theorem
Hall's + Compactn. Theorem
Dilworth's Theorem
-Posets-
Max antichain
≅
Min chain dec.
Ford-Fulkerson Algorithm
-W. Graphs-
Max Flow
≅
Min Cut

Conclusions and Work in progress

The main characteristics of our formalisation are:

- Use of standard definitions that simplify further extensions and applications;
- Closeness to pen-and-paper proofs, dissecting all minimal required steps in the assisted proof.

⇒ Exhibiting all minimal details is relevant to highlight to Math and CS students and professionals the relevance of mechanised proofs.

- Having a library of combinatorial theory is essential to specify and verify algorithms over graph structures and sets.
- The development is available as an input in the *Archive of Formal Proofs*:

  "Compactness Theorem for Propositional Logic and Combinatorial Applications"

Conclusions and Work in progress

Thank for your attention!