

# A PVS Library on the Infinitude of Primes (Extended Version)<sup>\*</sup>

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**Abstract.** This paper discusses the formalization in PVS of diverse proofs of the infinitude of primes. These proofs are developed using techniques from various areas of mathematics, including set theory, algebra, analysis, number theory, and topology. The availability of such a variety of proofs is helpful as a didactic resource, aiming to encourage mathematicians working in different areas to adopt interactive theorem provers as one of their everyday tools. The presented collection of formalizations follows the proofs selected by Erdős, Aigner, and Ziegler in their famous work “Proofs from THE BOOK,” namely those based on Fermat numbers, Mersenne numbers and algebraic structures, topological properties, and the analysis of harmonic series. The paper discusses the differences between informal proofs and mechanical formalization, highlighting the usefulness of distinguishing features of PVS to guide and facilitate the presented mechanization.

**Keywords:** Primes, Fermat Numbers, Mersenne Numbers, Harmonic Series, Theorem Proving, Algebraic Formalizations, PVS.

## 1 Introduction

Euclid’s proof of the infinitude of primes [4] is a classic and highly illustrative result. As the concept of primality is typically presented in introductory math courses, this proof offers an excellent example of approaching problems involving infinity. Over the years, many mathematicians, such as Paul Erdős, have provided new proofs of this result, each drawing from different areas of mathematics. These proofs are not only valuable for showcasing the tools offered by such diverse fields, but they also serve as a reminder that mathematics is a profoundly

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interconnected discipline, where concepts and techniques from diverse branches often come together to solve fundamental problems.

In the context of formalizing mathematical knowledge, proof assistants offer invaluable tools to ensure rigor and correctness. They provide a structured and reliable approach to formalizing and verifying logical reasoning, ensuring that the proof is free of errors, ambiguities, and gaps. This work presents five alternative proofs of the infinitude of primes using the Prototype Verification System (PVS) [26]. These formalizations explore various proof techniques derived from different areas of mathematics, including algebra, number theory, topology, and analysis. Each proof is constructed carefully to ensure logical consistency and rigor. The proofs are derived from those in “Proofs from THE BOOK” by Martin Aigner and Günter Ziegler [1], which offer six different proofs. Euclid’s classical proof, the first in the mentioned book, is omitted here as it is already part of the NASA PVS Libraries, *NASALib*<sup>4</sup>. The presented mechanization relies on results from diverse libraries, including those from NASALib and the PVS prelude. NASALib offers valuable abstractions for mathematical structures, including sets, groups, and Cartesian products.

Notably, this work does not assume the infinitude of primes beforehand, as circular reasoning is not accepted by proof assistants such as PVS. This kind of circularity can arise inadvertently in manual theorem proving when using a result much stronger than necessary. A notable example is the use of the Gödel Completeness Theorem [18] to prove the Compactness Theorem. In “Proofs from THE BOOK,” notation such as  $p_1, p_2, p_3, \dots$  is used for prime enumeration, but notice that this type of notation assumes the infinitude of primes beforehand.

One key aspect of this study is the identification and correction of notational errors and *informalities* in “Proofs from THE BOOK.” PVS’s robust type system helped to highlight and address these flaws, ensuring the proofs are precise and rigorous. Moreover, this work emphasizes the educational value of using PVS to formalize mathematical proofs. By breaking down the proofs into step-by-step procedures, this work not only demonstrates various formal proof techniques but also serves as a pedagogical resource. It offers readers the opportunity to learn how to structure and validate proofs within a proof assistant, fostering a deeper understanding of formal methods in mathematics. Thus, the mechanization of these proofs serves both as a study of mathematical reasoning and as a guide to using proof assistants effectively in diverse mathematical contexts.

## 1.1 Related Work

A significant number of the needed theorems for fields such as algebra, number theory, analysis, and topology are already available as PVS formalizations in NASALib [7, 19, 23]. These theorems were imported when the code was initially set up, which significantly streamlined the work. This allows for a solid foundation, avoiding the need to prove basic results and instead focusing on more advanced or specific aspects of the problem at hand.

<sup>4</sup> [https://github.com/nasa/pvslib/blob/master/numbers/infinite\\_primes.pvs](https://github.com/nasa/pvslib/blob/master/numbers/infinite_primes.pvs).

Euclid’s classic proof of the infinitude of primes has been formalized in various proof assistants, each presenting different approaches. One notable collection of such formalizations can be found in the “Formalizing 100 Theorems” project [31], which references formalizations on eleven different proof assistants. The usual strategies employed in these formalizations often revolve around two key techniques. One approach uses the product of primes plus one variant of Euclid’s proof, as seen in proofs formalized in systems like Naproche [21] and the NASALib itself. The other approach employs a factorial plus one method, which is used in the Isabelle/HOL and Coq proofs.

In addition to the classical Euclid’s proof of the infinitude of primes, other proofs have been developed using different proof assistants, such as those found in Isabelle. Such proofs are Fürstenberg’s topological proof [13] and another involving the zeta function [12]. The former formalization imports the Isabelle HOL theories for reals, number theory, and analysis, and in addition to the formalization of Fürstenberg’s proof, includes also proof of properties of Fürstenberg’s topology. However, the topology-based proof of the infinitude of primes is elementary to formalize, as it relies on fewer mathematical structures compared to other proofs in “Proofs from THE BOOK” (“THE BOOK,” for short). Indeed, Fürstenberg’s proof requires essentially few basic notions of topological spaces, set theory, and number theory, leaving less room for alternative approaches. As a result, the existing formalizations differ primarily in how they are handled by different proof assistants, rather than in the structure of the proof itself. For didactic matters, the current formalization of Fürstenberg’s proof imports the topology library and the minimum necessary notions from NASALib. On the other hand, the proof using the zeta function employs the Euler Product and the Cauchy Equality. It is presented in “THE BOOK,” and also covered in the current formalization. The formalization in [12] presents a significant divergence regarding “THE BOOK.” It employs a more complex approach that utilizes the analytic continuation of the zeta function and then leverages the divergence at  $s = 1$  to prove the infinitude of primes. In contrast, the current formalization follows a pedagogical approach, also importing the minimum necessary PVS libraries on series and reals. However, some non-trivial results assumed in “THE BOOK,” such as the Cauchy Equality, were mechanized to obtain a complete formalization of this proof.

While the primary focus of this paper is on the first topic of “THE BOOK,” which addresses the infinitude of primes, it is also worth noting that there are other formalizations in “THE BOOK” beyond this first topic. These include proofs of the irrationality of certain numbers [27] and Fermat’s two-square theorem [9].


## 1.2 Main Contributions

The main contributions of this work are:

- The formalization in PVS of five additional proofs for the infinitude of primes, which can be presented as applications of the results from various NASALib’s libraries, such as `ints`, `algebra`, `analysis`, and `topology`.

- The discussion and formalization of omitted details in “THE BOOK.”
- A new approach for the standard prime factorization theorem in NASALib and general structure specification.
- Several improvements in the `algebra` library, such as the  $\mathbb{Z}/p\mathbb{Z}$  coset manipulation and type-checking related problems.
- Minor improvements in the manipulation of integer expressions in PVS, especially related to the *gcd* function.

### 1.3 Organization

Section 2 sketches the informal proofs that guide the formalization presented in this paper. Section 3 discusses aspects of the formalizations, focusing on the two more interesting proofs in terms of the usage of distinguishing typing features provided by PVS and the level of difficulty involved in their mechanical verification. Section 4 concludes the paper by providing some final remarks, also providing quantitative data, and discussing possible lines of future work. The paper includes hyperlinks to specific points of the formalization using the symbol . The complete formalization is available at [https://github.com/nasa/pvslib/tree/master/ints/inf\\_primes](https://github.com/nasa/pvslib/tree/master/ints/inf_primes). An extended version of this work includes more detailed information on the proofs and the PVS formalization [25].

## 2 Brief Description of the Informal Proofs

This section provides a brief description of the proofs addressed in the presented formalization. In the following, the set of prime numbers is denoted by  $\mathbb{P}$ .

### 2.1 Fermat Numbers

The second proof detailed in [1] uses number theory [20]. More precisely, it uses the infinitude of the Fermat numbers [28]. The Fermat numbers are of the form:

$$F_n = 2^{2^n} + 1, \text{ where } n \in \mathbb{Z}_{\geq 0}.$$

The main idea guiding the proof is to show that Fermat numbers are pairwise relatively prime. In other words, each Fermat number must have at least one distinct prime divisor. Since it is possible to find infinitely many Fermat numbers, it follows that there must be infinitely many prime numbers. Since NASALib and the PVS prelude provide a strong set of theorems in number theory, this proof turned out to be one of the shortest.

### 2.2 Mersenne Numbers

The third proof uses the Mersenne numbers [28], which are defined as  $M_n = 2^n - 1$ ,  $n \in \mathbb{Z}_{\geq 0}$ . In this proof,  $n$  is restricted to the set of prime numbers and is denoted by  $p$ . The main idea of the proof is to show that there exists a prime

divisor  $q$  of  $M_p$  such that  $q$  is greater than  $p$ . If there were finite primes, there must exist a maximum prime  $p_{max}$ . This is a contradiction since one can find a greater prime in the set of divisors of  $M_{p_{max}}$ .

The approach followed in “Proofs from THE BOOK” is based on abstract algebra, the application of Lagrange’s Theorem [22], and the fact that  $\mathbb{Z}_q \setminus \{0\}$  is a group under multiplication. The proof is structured as described below.

1. Let  $p$  be an arbitrary prime and  $q$  be one prime factor of  $M_p = 2^p - 1$ . Notice that  $q$  must be odd since  $2^p - 1$  is odd.
2. Since  $q$  divides  $2^p - 1$ , this implies that  $2^p \equiv 1 \pmod{q}$ . The number  $p$  is a prime; thus, it must be the order of the element  $\bar{2}$  in  $\mathbb{Z}_q \setminus \{0\}$ . Otherwise, there would be  $r \in \mathbb{N}, 1 < r < p$ , which divides  $p$ .
3. An element  $\bar{a} \in \mathbb{Z}_q \setminus \{0\}$  of order  $n$  generates a subgroup  $\langle \bar{a} \rangle = \{\bar{a}^i : i \in \mathbb{Z}_{\geq 0}\}$  with cardinality  $|\langle \bar{a} \rangle| = n$ . By applying Lagrange’s Theorem,  $|\langle \bar{2} \rangle| = p$  divides  $|\mathbb{Z}_q \setminus \{0\}| = q - 1$ .
4. Assume there exists a maximum prime  $p_{max}$ . Thus, there exists  $q \in \mathbb{P}$  such that  $p_{max} \mid q - 1$ . Consequently,  $p_{max} \leq q - 1$ , and  $p_{max} < q$ , which is a contradiction. Therefore, there are infinitely many primes.

### 2.3 Euler Product Formula and Cauchy Equality

The structure of the manual proof can be divided into the following steps.

1. Let  $\pi(n)$  be the prime-counting function that counts the number of prime numbers smaller than or equal to  $n$ . Suppose there exists an enumeration of  $\mathbb{P}$  in increasing order.
2. The harmonic numbers can be underestimated with natural logarithms as

$$\log(n) \leq H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

3. The product of a geometric series of inverse prime numbers less than or equal to  $n$  is equal to another series that contains every  $\frac{1}{k}$  from  $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ :

$$H_n \leq \prod_{i=1}^{\pi(n)} \sum_{k=0}^{\infty} \frac{1}{p_i^k} = \sum_{\substack{k \in \mathbb{Z}_{\geq 0}, \\ k=1 \vee \exists p \in \mathbb{P}, \\ (p \leq n \wedge p \mid k)}} \frac{1}{k}.$$

4. For each prime number  $p_i$ , the geometric series  $\sum_{k=0}^{\infty} \frac{1}{p_i^k}$  converges to  $\frac{p_i}{p_i - 1}$ .

Also,  $p_i \geq i + 1$ , which implies that  $\frac{p_i}{p_i - 1} \leq \frac{i + 1}{i}$ . Consequently,

$$\prod_{i=1}^{\pi(n)} \sum_{k=0}^{\infty} \frac{1}{p_i^k} = \prod_{i=1}^{\pi(n)} \frac{p_i}{p_i - 1} \leq \prod_{i=1}^{\pi(n)} \frac{i + 1}{i} = \pi(n) + 1.$$

5. By arranging inequalities,  $\log(n) \leq \pi(n) + 1$ . Since the natural logarithmic function is strictly increasing, the sequence generated by the  $\pi$  function diverges, which means that  $\mathbb{P}$  is infinite.

## 2.4 Fürstenberg's Topological Proof

Hillel Fürstenberg introduced this elegant proof as a 12-line note in the section on Mathematical Notes of the American Mathematical Monthly in 1995 [16]. This non-traditional approach builds a topology [24] on integer numbers. The structure of this proof can be divided into the following parts.

1. Given  $a, b \in \mathbb{Z}$ , where  $b > 0$ , define the family of sets  $N_{a,b} = \{a + bn : n \in \mathbb{Z}, b > 0\}$ .
2. A set  $O \subseteq \mathbb{Z}$  is called *open* whether  $O = \emptyset$  or for every element  $a \in O$ , there exists some  $b \in \mathbb{Z}, b > 0$  with  $N_{a,b} \subseteq O$ . As usual in topology, a *closed* set is defined as the complement of an open set in  $\mathbb{Z}$ .
3. By definition, the union of two open sets  $O_1 \cup O_2$  is another open set. Also, the intersection of two open sets is also an open set: if  $a \in O_1 \cap O_2$ , thus there exist  $b_1 > 0$  and  $b_2 > 0$ , such that  $N_{a,b_1} \subseteq O_1$  and  $N_{a,b_2} \subseteq O_2$ ; consequently,  $N_{a,b_1 b_2} \subseteq O_1 \cap O_2$ . Therefore, such open sets induce a well-defined topology.
4. For any  $a, b \in \mathbb{Z}, b > 0$ ,  $N_{a,b}$  is open. Also, notice that  $N_{a,b} = \mathbb{Z} \setminus \bigcup_{i=1}^{b-1} N_{a+i,b}$ .

Since  $N_{a,b}$  is the complement of the open set  $\bigcup_{i=1}^{b-1} N_{a+i,b}$ , thus  $N_{a,b}$  is a closed set.

5. If  $O$  is a nonempty open set then  $O$  is infinite, since  $N_{a,b} \subseteq O$  for some  $b > 0$ .
6. Every  $n \in \mathbb{Z} \setminus \{-1, 1\}$  has a prime divisor  $p$ , which implies that  $n \in N_{0,p}$ . Consequently,  $\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in \mathbb{P}} N_{0,p}$ .
7. If  $\mathbb{P}$  is finite, then  $\mathbb{Z} \setminus \{-1, 1\}$  is a closed set since it is a finite union of closed sets, as pointed out above. Consequently,  $\{-1, 1\}$  is an open set, which is a contradiction since all open sets in this topology are infinite.

## 2.5 Prime Reciprocal Harmonic Series

Paul Erdős originally proved the sixth and last proof in the 20th century [14] and can be viewed as inspired by the proof found in Section 2.3. The main idea is to consider another series of reciprocal numbers, but instead of using the positive integers, the prime numbers are used, i.e.,  $\sum_{i=1}^n \frac{1}{p_i}$ . As a finite summation of numbers converges, if this series diverges, the set of primes must be infinite.

In this proof, the set of primes is divided into two types: the *Small* primes, which are smaller or equal to a prime  $p_k$ , and *Big* primes, the remaining ones. From this classification, other sets are defined:  $N(n)$ , the set of positive numbers less than or equal to  $n$ ;  $N_s(n, k)$ , the numbers from  $N(n)$  with only *Small* prime divisors;  $N_b(n, k)$  the numbers from  $N(n)$  with at least one *Big* prime divisor. It can be shown that  $N(n) = N_s(n, k) \cup N_b(n, k)$ .

1. Consider a prime enumeration  $p_i$  and suppose that the series  $\sum_{i=1}^N \frac{1}{p_i}$  converges. Therefore exists a  $\kappa$  such that  $\sum_{i=\kappa+1}^{\infty} \frac{1}{p_i} < \frac{1}{2}$ .
2. Define  $N_{div}(d, n)$  the subset of  $N(n)$  whose elements are multiples of  $d \in \mathbb{N}, d \geq 1$ . It can be proven that  $|N_{div}(d, n)| = \lfloor \frac{|N(n)|}{d} \rfloor = \lfloor \frac{n}{d} \rfloor$ . Noticing that  $N_b(n, k)$  is the union of all  $N_{div}(p_i, n)$ , where  $i > k$ , one can estimate the size of  $N_b(n, \kappa)$  by:

$$|N_b(n, \kappa)| \leq \sum_{i=\kappa+1}^{\infty} \left\lfloor \frac{n}{p_i} \right\rfloor \leq \sum_{i=\kappa+1}^{\infty} \frac{n}{p_i} < \frac{n}{2}.$$

3. An element  $m \in N_s(n, k)$  can be written as  $m = a \cdot b$ , where  $a, b \in N_s(n, k)$ ,  $a$  is a square-free part of  $m$ , and  $b$  is a perfect square of an element of  $N_s(n, k)$ . From this observation, two other sets are defined:  $S_{free}(n, k)$ , composed of all elements  $a$ , and  $S_{div}(n, k)$ , composed of all elements  $b$ . With these considerations, the size of  $N_s(n, k)$  is estimated:


$$|N_s(n, k)| \leq |S_{free}(n, k) \times S_{div}(n, k)| = |S_{free}(n, k)| \cdot |S_{div}(n, k)|.$$

4. Since  $m = a \cdot b$  for all  $m \in N_s(n, k)$ , the number of elements of  $S_{div}(n, k)$  can be estimated by setting  $a = 1$  and using the definition of  $b$ , i.e.  $b = r^2$  for  $r \in N_s(n, k)$ . Finding the size of  $S_{div}(n, k)$  turns into a problem of counting the number of valid  $m = r^2$ . Noticing that  $N_s(n, k) \subseteq N(n)$ , one can establish:  $|S_{div}(n, k)| \leq \sqrt{|N_s(n, k)|} \leq \sqrt{|N(n)|} = \sqrt{n}$ .
5. An element of  $S_{free}(n, k)$  is of the form  $m = p_1^{\epsilon_1} \cdot p_2^{\epsilon_2} \cdots p_k^{\epsilon_k}$ , where  $\epsilon_i \in \{0, 1\}$ . Consequently,  $|S_{free}(n, k)| \leq 2^k$ .
6. Since  $N(n) = N_s(n, k) \cup N_b(n, k)$ , for every  $k$ , one concludes that:

$$|N(n)| \leq |N_s(n, \kappa)| + |N_b(n, \kappa)| < 2^\kappa \sqrt{n} + \frac{n}{2}.$$

7. In particular, if  $n = 2^{2\kappa+2}$  then  $|N(n)| < 2^{2\kappa+2} = n$ , which is a contradiction, since  $|N(n)| = n$ . Therefore, the original consideration of the convergence of a series of prime reciprocals must be false. That is only possible if there are infinitely many primes.

### 3 Description of the Formalization

Only the formalizations of the proofs based on Mersenne Numbers and on the Euler Product Formula are discussed here, as they required significantly more effort than the traditional proofs. The formalization based on the Harmonic Prime Reciprocal Series  presented the most significant divergences from the informal proof. On the contrary, it was possible to develop a formalization fairly close to

the manual proofs for the ones based on Fermat Numbers [15] and Fürstenberg’s topological arguments [16].

Before diving into the details, it is worth noticing the main building blocks on which this effort is founded. In addition to the PVS prelude and basic NASALib libraries such as `set` and `structures`, the presented formalization leverages specialized results from the NASALib libraries `algebra`, `topology`, `series`, and `analysis`. Notably, the concepts of topological spaces and relations between open and closed sets were taken from the library `topology`. Some properties about limits and integrals were imported from the `analysis` library. The `series` library provides properties about the convergence of (infinite) series. Finally, from the `algebra` library, results related to (finite) groups and cosets were used. As an original contribution to these libraries, several results were added, such as a reformulation of the Fundamental Theorem of Arithmetic and a version of the Cauchy Product Theorem, among others.

### 3.1 Mersenne Numbers

The first design decision addressed how to specify the multiplicative group  $\mathbb{Z}_p \setminus \{0\}$ , where  $p$  is a prime number. Although there is a specification for the ring  $\mathbb{Z}/n\mathbb{Z}$  [17], there exists no direct implementation for the multiplicative group  $\mathbb{Z}/n\mathbb{Z} \setminus \{n\mathbb{Z}\}$ . The group-related theorems that were applied belong to theories that rely on the following assumption: the set of all elements of an abstract type  $T$  must satisfy a group predicate [18]. In other words, the importation of these theories introduces a *Type Correctness Condition* (TCC) automatically generated by the system, which is a proof obligation for checking whether the type  $T$  consists of a complete set of elements forming a group. This is not a direct application of the lemma `Zp_prime_is_field` [19], already in NASALib, stating that  $\mathbb{Z}/p\mathbb{Z}$  is a field when  $p$  is a prime number and thus, that  $\mathbb{Z}/p\mathbb{Z} \setminus \{p\mathbb{Z}\}$  forms a group under multiplication. Indeed, the specification of `field` in the theory `field_def` [20], from a division ring [21], gives the flexibility of considering the set of cosets of  $n\mathbb{Z}$  in  $\mathbb{Z}$  as a parameter in the lemma `Zp_prime_is_field` [19], without excluding the identity for addition  $n\mathbb{Z}$ . To use the results in theory `finite_groups`, it was necessary to specify the type `nz_coset(n)` [22] and then prove that it satisfies the group properties when  $n$  is a prime number [23]. Roughly, it could be done by using the lemma `Zp_prime_is_field` combined with enough expansions of the definition of `group?(nz_coset(n))(Z(n))` in the lemma `nz_prime_is_group`.

Still, some TCCs appeared during the manipulation of elements of type `nz_coset(p)`; for this reason, additional utility lemmas were proved and separated in the `ring_zn_extra.pvs` [24] file, as they could be used in more general situations. The content of this file ranges from lemmas of equivalence of the operations in  $\mathbb{Z}_n$  and  $\mathbb{Z}/n\mathbb{Z}$  to some direct ring properties, such as product and summation closure, and the characteristic of the ring  $\mathbb{Z}_p$  being  $p$ .

It is worth mentioning that some type-related proofs can be avoided; instead of using generic definitions such as the power function specified in the group file, it is possible to define a specialized function for handling this new `nz_coset` type. This could be done by forcing the type to be `nz_coset` instead of the



PVS-deduced coset type. For example, the signature of the power function was restricted to  $pow : \mathbb{Z}/p\mathbb{Z} \setminus \{p\mathbb{Z}\} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}/p\mathbb{Z} \setminus \{p\mathbb{Z}\}$  instead of using the more general version  $pow : \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}/p\mathbb{Z}$ .

After the explanation of these preliminaries, the actual proof of the infinitude of prime numbers can be finally discussed.

**Lemma 1.** *If  $d$  is a divisor of  $M_p$  where  $p \in \mathbb{P}$ , then  $d$  is odd.*

*Proof.* Since  $p$  is a prime number,  $p \geq 2$ , implying that  $M_p = 2 \cdot 2^{p-1} - 1$  is odd. Suppose that  $d$  is even. Since it is a divisor of  $M_p$ ,

$$M_p = d \cdot k_1, k_1 \in \mathbb{Z}.$$

By the evenness of  $d$

$$M_p = 2 \cdot k_2 \cdot k_1, k_2 \in \mathbb{Z}.$$

This is a contradiction since  $M_p$  is odd.

**Lemma 2.** *Let  $q, p \in \mathbb{P}$ , where  $q$  is a divisor of  $M_p$ , then*

$$(2 + q\mathbb{Z})^{q-1} = 1 + q\mathbb{Z}.$$

*Proof.* By Lemma 1,  $q$  is an odd number since  $q$  is a prime  $q \geq 3$ . By Fermat's Little Theorem,

$$2^{q-1} \equiv 1 \pmod{q}, \text{ which implies that } (2 + q\mathbb{Z})^{q-1} = 1 + q\mathbb{Z}.$$

The last equation comes from the ring isomorphism  $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ .

In the PVS specification, the equivalence in the modular arithmetic formulation and quotient ring formulation was proved directly. It was also necessary to adapt Fermat's Little Theorem to the requirements in the proof: it was specified in the  $a^p \equiv a \pmod{p}$  form, not in the  $a^{p-1} \equiv 1 \pmod{p}$  form.

**Lemma 3.** *Let  $q, p \in \mathbb{P}$ , where  $q$  is a divisor of  $M_p$ , then*

$$(2 + q\mathbb{Z})^p = 1 + q\mathbb{Z}.$$

*Proof.* Since  $q$  divides  $M_p$ ,

$$M_p \equiv 0 \pmod{q}, \text{ which implies}$$

$$2^p - 1 \equiv 0 \pmod{q}, \text{ which is equivalent to}$$

$$2^p \equiv 1 \pmod{q}.$$

Using the isomorphism  $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ .

$$(2 + q\mathbb{Z})^p = 1 + q\mathbb{Z}.$$

**Theorem 1.** *There are infinitely many primes.*

*Proof.* Suppose there exists a finite number of prime numbers; then there should exist a maximum prime  $p_{max}$ . Let  $q \in \mathbb{P}$  be the divisor of  $M_{p_{max}}$ . Using Lemma 3,


$$(2 + q\mathbb{Z})^{p_{max}} = 1 + q\mathbb{Z}.$$

In particular, from the definition of order, it follows that  $ord(2 + q\mathbb{Z}) \mid p_{max}$ , but that is only possible if  $ord(2 + q\mathbb{Z}) = 1$  or  $ord(2 + q\mathbb{Z}) = p_{max}$ . If  $ord(2 + q\mathbb{Z}) = 1$ , then  $2 + q\mathbb{Z} = 1 + q\mathbb{Z}$ , which is not possible since  $q > 1$ . Therefore, it must be the case that  $ord(2 + q\mathbb{Z}) = p_{max}$ .

Using Lemma 2,

$$(2 + q\mathbb{Z})^{q-1} = 1 + q\mathbb{Z}.$$

Again, by the definition of order,  $ord(2 + q\mathbb{Z}) \mid q - 1$  and  $p_{max} \mid q - 1$ . Since a divisor is smaller than or equal to the number it divides,  $p_{max} \leq q - 1$ . More specifically,  $p_{max} < q$ . Therefore,  $q$  is a prime greater than the maximum prime, a contradiction.

It turns out that Lagrange’s Theorem was not necessary. In fact, if it had been used, it would have been necessary to prove additional lemmas on group orders; however, these proofs can be quite tedious. Instead, the following classical theorem was used: if an element  $a$  from a group  $G$  satisfies  $a^n = 1$  for some integer  $n$ , then  $ord(a)$  divides  $n$  .

Related to TCCs, since the definition of structures in the NASALib’s algebra library, such as rings, is built upon the group definition, and these in turn are based on monoids (and so on), type dependencies become an exhaustive issue. The problem arises because they require a significant number of TCCs. If such structures are imported naively, each new algebraic structure used in a proof could generate around five new TCCs. Consequently, there is room for improvement in the algebra library from various angles, such as through new utility theorems, new proof strategies (conservative extensions of the proof calculus provided by PVS), and possibly type judgments, which provide more information to the type checker. Nevertheless, the algebra library contains many powerful theorems, including classic results from group and ring theory, such as Lagrange’s Theorem, Sylow’s Theorems, and many others, some of which facilitated the presented work.

### 3.2 Formalization Based on Euler Product Formula and Cauchy Equality

The fourth proof in “THE BOOK” relies on analytic number theory [3]. As a side effect of Euler’s Formula [15], proved in the 18th century by Leonhard Euler, this proof has a deep connection to the Riemann zeta function [29]. The key idea is to demonstrate that the zeta function can be factored into a product over prime numbers. With this connection, the estimation for the number of primes can be as large as desired, confirming that primes are indeed infinite.

The Riemann-zeta function is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } s \in \mathbb{C}, \quad \text{Re}(s) > 1.$$

Euler's product formula, on the other hand, relates the primes in the following way:


$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{for } s \in \mathbb{C}, \quad \text{Re}(s) > 1.$$


Notice that, from the definition of the zeta function,  $s$  must have a real part greater than one. It turns out that this Euler product also works for  $s = 1$ , but the zeta function at this value tends to infinity, something that should not happen if there are finite primes.





In particular, it is possible to estimate the prime-counting function by the product of the primes according to the Euler formula, which by itself can be bounded using the natural logarithm function in the following way:

$$\log(n) \leq \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-1} \leq \pi(n) + 1.$$


where  $\log(n)$  is the natural logarithm function and  $\pi(n)$  is the function that counts the number of prime numbers less than or equal to a given number  $n$ .

As is typical in traditional number theory proofs, this proof heavily relies on concepts from analysis, such as limits and series, which are addressed in the NASALib Analysis library  [19].


**3.2.1 Prime Enumeration** The notation  $\mathbb{P} = \{p_1, p_2, p_3, \dots\}$  in [1] has the problem of assuming that the set of prime numbers  $\mathbb{P}$  is infinite beforehand, and the sequence should be undefined otherwise. For simplicity, in the specification, the starting index is zero, and undefined cases are set to zero. This means that if the prime numbers have an end at the  $n$ th value, then  $p_i = 0$  for  $i \geq n$ . Therefore,  $p_0 = 2$   $p_1 = 3$   $p_2 = 5$ . The proper definition of the prime sequence is given by a function  $\rho : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  . It remains to prove that for a subset of the domain,  $S \subseteq \mathbb{Z}_{\geq 0}$ , this function is an enumeration. For this purpose, some necessary properties are inductively formalized, such as:

1.   $\rho(i+1) > \rho(i) \vee \rho(i+1) = 0$ ;
2.   $\forall \rho(i), \rho(j) \in \mathbb{P}, \rho(i) = \rho(j) \Rightarrow i = j$ ;
3.   $\forall p \in \mathbb{P}, \exists i \in \mathbb{Z}_{\geq 0}, \rho(i) = p$ ;
4.  Let  $i, n \in \mathbb{Z}_{\geq 0}, i < \pi(n) \Rightarrow \rho(i) \in \mathbb{P}$ .



The  $\rho$  function is indeed an enumeration for a subset of the domain. If there are infinitely many primes, all primes will appear in ascending order in the domain  $\mathbb{Z}_{\geq 0}$ . Otherwise, for the domain  $S = \{n \in \mathbb{Z}_{\geq 0} : n < \pi(p_{max})\}$ , all primes will also appear in ascending order, and in its complement,  $\mathbb{Z}_{\geq 0} \setminus S$ , the function will be zero.

In the following proofs, it will be necessary to use the Fundamental Theorem of Arithmetic [17]. This theorem is in NASALib , but specified generically: any natural greater than one can be written as a product of a non-decreasing sequence of primes, for example,  $360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$ . For the current purposes, a specialized version of this theorem is used, stating that any natural number greater than one can be written as a product of an increasing sequence of powers of primes, for instance,  $360 = 2^3 \cdot 3^2 \cdot 5$ .

Because a prime enumeration was already specified, it can be used to specify the prime powers in sorted form. Still, knowing beforehand that there are infinitely many primes, one should be tempted to describe the Fundamental Theorem as the existence of the infinite product, with large enough terms having exponent zero, such as  $360 = \rho(0)^3 \cdot \rho(1)^2 \cdot \rho(2)^1 \cdot \rho(3)^0 \cdot \rho(4)^0 \dots$ . However, this would lead to the same mistake of circularly assuming that there are infinitely many primes. Therefore, a new version of the Fundamental Theorem was formalized.



Given a family of sets  $E_p = \{n \in \mathbb{Z} : \exists k \in \mathbb{Z}_{\geq 0}, n = p^k\}$ , where  $p \in \mathbb{P}$ , the set  $D_n$  can be defined as the finite Cartesian product shown below .

$$D_n = \bigtimes_{i=0}^{\pi(n)-1} E_{\rho(i)}.$$


Thus, the Fundamental Theorem can be rewritten as the existence of a unique element  $(\rho(0)^{\epsilon_0}, \rho(1)^{\epsilon_1}, \dots, \rho(\pi(n)-1)^{\epsilon_{\pi(n)-1}}) \in D_n$ , such that the product of its entries  equals  $n$  for every  $n \in \mathbb{Z}, n > 1$ ; i.e., .

$$n = \prod_{i=0}^{\pi(n)-1} \rho(i)^{\epsilon_i}.$$

Since the greatest prime divisor of a number is the number itself, the upper limit of the product,  $\pi(n) - 1$ , guarantees that all prime divisors will appear in the product.

It is worth mentioning that the definition of prime enumeration and prime factorization is reused for the formalization based on prime reciprocal series (Subsection 2.5); because of that, these proofs, alongside another general purpose  $\rho$  function manipulation, were separated to a new file called `prime_enum` . Additionally, using this new framework for the proof of the Fundamental Theorem of Arithmetic, the application of lemmas related to integers was useful. Among these lemmas, some properties related to the  $\gcd$  function were recently added to NASALib in the file `number_util.pvs` .

**3.2.2 A Few Inequalities** For the completion of the proof, a few inequalities must be proven, starting from a classic one.


**Lemma 4.**   $\forall n \in \mathbb{Z}_{\geq 0}, \log(n) \leq H_n.$

*Proof.* This can be done by considering the inequality  $\frac{1}{x} \leq \frac{1}{k}$  for  $x \in [k, k+1]$ , and the inequalities for finite summations of integrations:

$$\log(n+1) = \int_1^{n+1} \frac{1}{x} dx = \sum_{k=1}^n \int_k^{k+1} \frac{1}{x} dx \leq \sum_{k=1}^n \int_k^{k+1} \frac{1}{k} dx.$$

Then  $\log(n+1) \leq \sum_{k=1}^n \frac{1}{k} = H_n$ , and since  $\log$  is an increasing function,  $\log(n) \leq H_n$ .

Despite being well known, this inequality was not explicitly stated in NASALib, but all its prerequisites had already been proven in the `analysis` library. This made its assisted proof relatively easy. The only minor problem was a TCC related to the integrability of each integral expression required in the proof; as the summation is applied over slices of the larger integral, it was necessary to ensure that everything is indeed integrable. However, lemmas for these steps were also in the files defining the logarithmic function and integral operations.


The next inequality uses two definitions of functions. At first glance, the defined functions appear to be different, but they are actually equivalent . Let  $n \in \mathbb{Z}_{\geq 0}$ ,  $n \geq 2$ ; the functions  $\xi$  and  $\mu$  are defined as:

$$\xi(n) = \prod_{i=0}^{\pi(n)-1} \sum_{k=0}^{\infty} \frac{1}{\rho(i)^k} \quad \text{and} \quad \mu(n) = \sum_{\substack{k \in \mathbb{Z}_{\geq 0}, \\ k=1 \vee \exists p \in \mathbb{P}, \\ (p \leq n \wedge p|k)}} \frac{1}{k}.$$


One thing to notice is that in  $\xi$ , there are divisions by  $\rho(i)$ , which can have zero value if one tries to use a nonexistent prime number, but as the product is taken from  $i = 0$  to  $i = \pi(n) - 1$ , using property 4, all  $\rho(i)$  values are primes. Even though it is not completely obvious, these two functions are indeed the same. Some non-trivial lemmas must be proven first to formalize this fact.

Given the Cauchy product [11], the product of two convergent series is another series.

$$\left( \sum_{n=0}^{\infty} a_n \right) \cdot \left( \sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} \sum_{k=0}^n a_{n-k} b_k. \quad (1)$$

This formula has the restriction of one of the series being absolutely convergent, but the series in the formalization is defined over positive numbers, making this restriction trivially valid. The last series in the formula can be flattened in such a way that it maintains its convergence, but to prove this, first, two other functions are defined.  Let  $n \in \mathbb{Z}_{\geq 0}$ , the functions  $\theta$  and  $\tau$  are defined as:

$$\theta(n) = \max \left( \left\{ k \in \mathbb{Z}_{\geq 0} : \frac{k(k+1)}{2} \leq n \right\} \right) \quad \text{and} \quad \tau(n) = n - \frac{\theta(n)(\theta(n)+1)}{2}$$

**Corollary 1.**  Let  $n, k \in \mathbb{Z}_{\geq 0}$ ,  $0 \leq k \leq n$ , then  $\theta(\frac{n(n+1)}{2} + k) = n$  and  $\tau(\frac{n(n+1)}{2} + k) = k$ .

*Proof.* Since  $0 \leq k$ ,  $\frac{n(n+1)}{2} \leq \frac{n(n+1)}{2} + k$ . Also, since  $k \leq n$ ,

$$\frac{n(n+1)}{2} + k \leq \frac{n(n+1)}{2} + n < \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}.$$


Therefore, by the definition of  $\theta$ , one must have  $\theta(\frac{n(n+1)}{2} + k) = n$ , implying that

$$\begin{aligned} \tau\left(\frac{n(n+1)}{2} + k\right) &= \frac{n(n+1)}{2} + k - \frac{\theta\left(\frac{n(n+1)}{2} + k\right) \left(\theta\left(\frac{n(n+1)}{2} + k\right) + 1\right)}{2} \\ &= \frac{n(n+1)}{2} + k - \frac{n(n+1)}{2} = k. \end{aligned}$$

**Corollary 2.** *Let  $c$  be a sequence, such that  $c_i = a_{(\theta(i)-\tau(i))} \cdot b_{\tau(i)}$ . For every  $n, k \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq n$ .*

$$c_{\frac{n(n+1)}{2} + k} = a_{n-k} \cdot b_k$$


*Proof.* Using the same argumentation as the Corollary 1, we have that for  $i = \frac{n(n+1)}{2} + k$ ,  $\theta(i) = n$  and  $\tau(i) = k$ , therefore  $c_i = a_{(\theta(i)-\tau(i))} \cdot b_{\tau(i)} = a_{n-k} \cdot b_k$

**Lemma 5.**  *Let  $n, k \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq n$ , then*

$$\sum_{k=0}^n a_{n-k} b_k = \sum_{k=\frac{n(n+1)}{2}}^{\frac{n(n+1)}{2} + n} a_{(\theta(k)-\tau(k))} \cdot b_{\tau(k)}.$$

*Proof.* Using the Corollary 1, it holds that  $\theta(\frac{n(n+1)}{2} + k) = n$  and  $\tau(\frac{n(n+1)}{2} + k) = k$ , therefore, by change of basis.

$$\begin{aligned} &\sum_{k=\frac{n(n+1)}{2}}^{\frac{n(n+1)}{2} + n} a_{(\theta(k)-\tau(k))} \cdot b_{\tau(k)} = \\ &\sum_{k=0}^n a_{\left(\theta\left(\frac{n(n+1)}{2} + k\right) - \tau\left(\frac{n(n+1)}{2} + k\right)\right)} \cdot b_{\tau\left(\frac{n(n+1)}{2} + k\right)} = \\ &\sum_{k=0}^n a_{n-k} b_k. \end{aligned}$$

**Lemma 6.**  *Let  $N \in \mathbb{Z}_{\geq 0}$ , then*

$$\sum_{n=0}^N \sum_{k=0}^n a_{n-k} b_k = \sum_{n=0}^{\frac{N(N+1)}{2} + N} a_{(\theta(n)-\tau(n))} \cdot b_{\tau(n)}.$$

*Proof.* By induction on  $N$ .

Case  $N = 0$ , by Corollary 1,  $\theta(0) = 0$  and  $\tau(0) = 0$ , therefore  $a_0 b_0 = a_{(\theta(0)-\tau(0))} \cdot b_{\tau(0)}$ .


Case  $N > 0$ , by i.h.

$$\begin{aligned} & \sum_{n=0}^{N+1} \sum_{k=0}^n a_{n-k} b_k = \\ & \sum_{k=0}^{n+1} a_{n-k} b_k + \sum_{n=0}^{\frac{N(N+1)}{2} + N} a_{(\theta(k)-\tau(k))} \cdot b_{\tau(k)}. \end{aligned}$$

Then, by Lemma 5, the last expression is equal to

$$\begin{aligned} & \sum_{k=\frac{(N+1)(N+2)}{2}}^{\frac{(N+1)(N+2)}{2} + N+1} a_{(\theta(k)-\tau(k))} \cdot b_{\tau(k)} + \sum_{n=0}^{\frac{N(N+1)}{2} + N} a_{(\theta(k)-\tau(k))} \cdot b_{\tau(k)} = \\ & \sum_{n=0}^{\frac{(N+1)(N+2)}{2} + N+1} a_{(\theta(k)-\tau(k))} \cdot b_{\tau(k)}. \end{aligned}$$

The next theorem requires formalizing one more lemma.

**Lemma 7.**  Let  $n \in \mathbb{Z}_{\geq 0}$ , there exist  $m, r \in \mathbb{Z}_{\geq 0}$ , with  $r \leq m$ , and  $n = \frac{m \cdot (m+1)}{2} + r$ .

*Proof.* By induction on  $n$ .

Case  $n = 0$ ,  $m = r = 0$  satisfy the equality.

Case  $n > 0$ . By i.h., if  $r = m$ ,

$$n+1 = \frac{m \cdot (m+1)}{2} + m+1 = \frac{(m+1)(m+2)}{2} = \frac{(m+1)[(m+1)+1]}{2} + 0.$$

Otherwise, if  $r < m$ ,  $n+1 = \frac{m \cdot (m+1)}{2} + (r+1)$ , and  $r+1 \leq m$ .

Now, the flattened version of the series can be shown to be equal to the original series.

**Theorem 2.** Let  $a_n$  and  $b_n$  be positive sequences and  $\sum_{n=0}^{\infty} \sum_{k=0}^n a_{n-k} b_k$  convergent.

Then  $\sum_{n=0}^{\infty} \sum_{k=0}^n a_{n-k} b_k = \sum_{k=0}^{\infty} a_{(\theta(k)-\tau(k))} \cdot b_{\tau(k)}$ .

*Proof.* It should be proved that  $\left| L - \sum_{k=0}^n a_{(\theta(k)-\tau(k))} \cdot b_{\tau(k)} \right| < \epsilon$ , for every real

$\epsilon > 0$ , for larger enough  $n \geq N$ , where  $L = \sum_{n=0}^{\infty} \sum_{k=0}^n a_{n-k} b_k$ .

By rewriting  $n$  as  $\frac{m \cdot (m+1)}{2} + r$ , where  $0 \leq r \leq m$  (Lemma 7), and using Lemma 6, the difference can be estimated as follows.


$$\begin{aligned}
& \left| L - \sum_{k=0}^{\frac{m \cdot (m+1)}{2} + r} a_{(\theta(k) - \tau(k))} \cdot b_{\tau(k)} \right| \\
&= \left| L - \sum_{k=0}^{\frac{m \cdot (m+1)}{2} + m} a_{(\theta(k) - \tau(k))} \cdot b_{\tau(k)} + \sum_{k=\frac{m \cdot (m+1)}{2} + r + 1}^{\frac{m \cdot (m+1)}{2} + m} a_{(\theta(k) - \tau(k))} \cdot b_{\tau(k)} \right| \\
&\leq \left| L - \sum_{k=0}^{\frac{m \cdot (m+1)}{2} + m} a_{(\theta(k) - \tau(k))} \cdot b_{\tau(k)} \right| + \left| \sum_{k=\frac{m \cdot (m+1)}{2} + r + 1}^{\frac{m \cdot (m+1)}{2} + m} a_{(\theta(k) - \tau(k))} \cdot b_{\tau(k)} \right| \\
&\leq \left| L - \sum_{k=0}^{\frac{m \cdot (m+1)}{2} + m} a_{(\theta(k) - \tau(k))} \cdot b_{\tau(k)} \right| + \left| \sum_{k=r+1}^m a_{m-k} \cdot b_k \right| \quad (\text{By Cor. 1}) \\
&\leq \left| L - \sum_{n=0}^m \sum_{k=0}^n a_{n-k} b_k \right| + \left| \sum_{k=r+1}^m a_{m-k} b_k \right| \quad (\text{By Lemma 6}) \\
&\leq \left| L - \sum_{n=0}^m \sum_{k=0}^n a_{n-k} b_k \right| + \sum_{k=0}^m a_{m-k} b_k \quad (\text{Since } a_n \text{ and } b_n \text{ are positive sequences}).
\end{aligned}$$


Since  $\sum_{n=0}^{\infty} \sum_{k=0}^n a_{n-k} b_k$  is convergent,  $\lim_{n \rightarrow \infty} \sum_{k=0}^n a_{n-k} b_k = 0$ , implying that for every  $\epsilon > 0$ , there exist  $N_1 \leq n, N_2 \leq n$ , such that

$$\left| L - \sum_{n=0}^m \sum_{k=0}^n a_{n-k} b_k \right| < \frac{\epsilon}{2} \quad \text{and} \quad \sum_{k=0}^m a_{m-k} b_k < \frac{\epsilon}{2}.$$

$$\text{Therefore, for } N = \max(N_1, N_2), \left| L - \sum_{k=0}^n a_{(\theta(k) - \tau(k))} \cdot b_{\tau(k)} \right| < \epsilon.$$

Since there was no previous specification of the Cauchy product in PVS, its formalization was essential to obtain a complete theory.

The series flattening process  was generalized for the product of more series.

**Lemma 8.**  Let  $n \in \mathbb{Z}_{>0}$ , then

$$\prod_{i=0}^{n-1} \sum_{k=0}^{\infty} a_i(k) = \sum_{\substack{k, j_l \in \mathbb{Z}_{\geq 0} \\ j_0 + j_1 + \dots + j_{n-1} = k}} \prod_{i=0}^{n-1} a_i(j_i).$$



*Proof.* By induction on  $n$ . For  $n = 1$ , this trivially says that  $\sum_{k=0}^{\infty} a_0(k) = \sum_{k=0}^{\infty} a_0(k)$ . For  $n > 1$ , by i.h.,

$$\prod_{i=0}^n \sum_{k=0}^{\infty} a_i(k) = \left( \sum_{k=0}^{\infty} a_n(k) \right) \cdot \left( \sum_{\substack{k, j_l \in \mathbb{Z}_{\geq 0} \\ j_0 + j_1 + \dots + j_{n-1} = k}} \prod_{i=0}^{n-1} a_i(j_i) \right).$$

By Cauchy product (Formula 1), the last expression is equal to

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{\substack{j_l \in \mathbb{Z}_{\geq 0} \\ j_0 + j_1 + \dots + j_n = m}} a_n(m - (j_0 + j_1 + \dots + j_{n-1})) \cdot \prod_{i=0}^{n-1} a_i(j_i) \\ &= \sum_{m=0}^{\infty} \sum_{\substack{j_l \in \mathbb{Z}_{\geq 0} \\ j_0 + j_1 + \dots + j_n = m}} a_n(j_n) \cdot \prod_{i=0}^{n-1} a_i(j_i) = \sum_{m=0}^{\infty} \sum_{\substack{j_l \in \mathbb{Z}_{\geq 0} \\ j_0 + j_1 + \dots + j_n = m}} \prod_{i=0}^n a_i(j_i) \\ &= \sum_{\substack{k, j_l \in \mathbb{Z}_{\geq 0} \\ j_1 + j_2 + \dots + j_n = k}} \prod_{i=0}^n a_i(j_i). \end{aligned}$$

The last equality uses the flattening process of Theorem 2.

The equality of functions  $\xi$  and  $\mu$ , used in the informal proof, can be obtained using Lemma 8 and the Fundamental Theorem of Arithmetic.

**Theorem 3.**  $\xi(n) = \mu(n)$

*Proof.* Using Lemma 8, we have

$$\prod_{i=0}^{\pi(n)-1} \sum_{k=0}^{\infty} \frac{1}{\rho(i)^k} = \sum_{\substack{k, j_l \in \mathbb{Z}_{\geq 0} \\ j_1 + j_2 + \dots + j_{(\pi(n)-1)} = k}} \prod_{i=0}^{\pi(n)-1} \frac{1}{\rho(i)^{j_i}}$$

Notice that every term of the right series is of the form


$$\frac{1}{\rho(0)^{\epsilon_1}} \cdot \frac{1}{\rho(1)^{\epsilon_2}} \cdots \frac{1}{\rho(m)^{\epsilon_m}}, \quad m = \pi(n) - 1$$

By the Fundamental Theorem of Arithmetic, this product results in a unique number  $\frac{1}{n}$ . In particular,  $\frac{1}{1}$  appears in the right series ( $\epsilon_i = 0$ ) and since we are

summing over all possibilities of exponents, every  $\frac{1}{n}$ , for a  $n$  that is divisible by some  $p \in \mathbb{P}, p \leq n$  is in the summation. As a result

$$\sum_{\substack{k, j_i \in \mathbb{Z}_{\geq 0} \\ j_1 + j_2 + \dots + j_{(\pi(n)-1)} = k}} \prod_{i=0}^{\pi(n)-1} \frac{1}{\rho(i)^{j_i}} = \sum_{\substack{k \in \mathbb{Z}_{>0}, \\ k=1 \vee \exists p \in \mathbb{P}, \\ (p \leq n \wedge p|k)}} \frac{1}{k}$$


However, in PVS, as this property was only needed in the next lemma, it was faster to associate each term directly in the next proof instead of stating this equality as a separate PVS lemma.


**Lemma 9.**  Let  $n \in \mathbb{Z}_{>0}$ ,  $H_n \leq \mu(n)$ .

*Proof.* From the definition of the  $\mu$  function, and since it is an absolutely convergent series, if for every  $\frac{1}{k}$ ,  $1 < k \leq n$ , there exists a prime  $p \mid k$ ,  $p \leq n$ , the series can be ordered as:

$$\mu(n) = \sum_{k=1}^n \frac{1}{k} + \sum_{\substack{k \in \mathbb{Z}_{>n}, \\ k=1 \vee \exists p \in \mathbb{P}, \\ (p \leq n \wedge p|k)}} \frac{1}{k}.$$

Which trivially results in  $H_n \leq \mu(n)$ . To conclude, since  $1 < k \leq n$  and a divisor is less than or equal to the number it divides, all prime divisors of  $k$  satisfy the inequality  $p \leq k$ . Therefore, the maximal prime divisor of  $k$ , say  $p$ , is such that  $p \leq n$ .

For the PVS formalization, such a series rearrangement needed to be expressed in a more explicit form; for that reason, a theory called **sequence\_extra**  was included, in which a constructive specification of the function that orders by common summed values is given.

**Lemma 10.**  Let  $n, i \in \mathbb{Z}_{\geq 0}$ , for  $i < \pi(n)$ ,  $\frac{\rho(i)}{\rho(i)-1} \leq \frac{i+2}{i+1}$ .


*Proof.* Notice that  $\frac{\rho(i)}{\rho(i)-1} \leq \frac{i+2}{i+1} \iff 1 + \frac{1}{\rho(i)-1} \leq 1 + \frac{1}{i+1} \iff i+1 \leq \rho(i)-1 \iff i+2 \leq \rho(i)$ .

This is proved by induction. For  $i = 0$ ,  $0 + 2 \leq 2$ ; for  $i > 0$ , by i.h.,  $i + 1 \leq \rho(i - 1)$ , which implies that  $i + 2 \leq \rho(i - 1) + 1$ . By the property 4,  $\rho(i) \neq 0$ , and using Lemma 1, it can be shown that  $\rho(i - 1) < \rho(i)$ . Since  $\rho(i - 1)$  is an integer,  $\rho(i - 1) + 1 \leq \rho(i)$ ; therefore,  $i + 2 \leq \rho(i)$ .

**Lemma 11.**   $\xi(n) \leq \pi(n) + 1$ .

*Proof.* Since the geometric series has a closed form,  $\xi(n)$  can be simplified as shown below.

$$\prod_{i=0}^{\pi(n)-1} \sum_{k=0}^{\infty} \frac{1}{\rho(i)^k} = \prod_{i=0}^{\pi(n)-1} \frac{\rho(i)}{\rho(i)-1} \leq \prod_{i=0}^{\pi(n)-1} \frac{i+2}{i+1}.$$

The last inequality is obtained using Lemma 10. Notice that the last expression is a telescoping product , therefore,  $\prod_{i=0}^{\pi(n)-1} \frac{i+2}{i+1} = \frac{(\pi(n)-1)+2}{0+1} = \pi(n)+1$ .

**Theorem 4.**  *There are infinitely many primes.*

*Proof.* Composing the inequalities from Lemmas 4, 11 and 9, one obtains  $\log(n) \leq \xi(n) = \mu(n) \leq \pi(n) + 1$ . Since the logarithm is a strictly increasing function, there is no maximum  $\pi(n)$  value.

## 4 Conclusion and Future Work


One of the goals of the presented PVS library is to demonstrate to mathematicians the potential of interactive theorem provers in formalizing complex mathematical concepts, showing that substantial and technically intricate proofs can be rigorously verified using computer software. The complete formalization of distinct proofs of the infinitude of primes from the renowned book “Proofs from THE BOOK” contributed significantly to the development of a rich and mathematically diverse library to attract the interest of researchers from various branches of mathematics. This represents a significant step forward regarding other libraries intended for algebraists and computer scientists (e.g., [2, 5, 6, 8]).

Table 1 shows a quantitative overview of the formalization effort.

Table 1: Quantitative data.

PVS theory	Formulas	TCCs	Specification Size (.pvs lines)	Proof Commands (.prf lines)	Main Dependencies				
					Prime enum	Cauchy Product	series	Topology (NASALib)	Algebra (NASALib)
Fermat	17	8	75	981					
Mersenne	28	17	112	2568					
Euler	39	28	112	3408	✓	✓	✓		
Fürstenberg	19	2	115	1822				✓	
Erdős	71	35	273	8117	✓		✓		
Additional Theories Quantitative Data									
PVS theory			Formulas	TCCs	Specification Size		Proof Commands		
Primes enumeration			65	37	212		4574		
Cauchy product formula			21	10	111		2302		
Series extra			23	8	109		1610		
Others			89	51	419		5125		

This work highlights essential differences between the original proofs and their mechanically checked counterparts. One of these differences is the way in which primes are initially enumerated in the pen and paper proofs, where no

consideration is given regarding the possibility of the existence of a maximum number of primes. To address such an imprecision, an enumeration function was defined in PVS, avoiding the initial implicit assumption on the infinitude of primes, and thus ensuring a rigorous foundation for the required adaptation of the Fundamental Theorem of Arithmetic. This result can be found in the file `prime_enum` . Furthermore, the rich typing system supported by PVS played a crucial role in highlighting the importance of distinguishing between the different algebraic structures at play. In particular, for the proof based on Mersenne numbers, the type system helped clarify the relationships between the different structures involved. Also, although Lagrange’s theorem is not used, the formalization leveraged a result about group orders, proving that the order of any group element satisfying a particular condition divides a given integer.

Another key difference between the proofs in “THE BOOK” and their formalizations arises when the Cauchy product is used to prove the Euler product. In “THE BOOK,” the connection between the Euler product and the harmonic series is somewhat informal, which required improving the rigorousness of the proof as part of the formalization effort. Since the Cauchy product was not included in NASALib, its incorporation benefits both the analysis and the Series library.

Additionally, the characteristic PVS features and existing PVS libraries were crucial in guiding the formalization effort, particularly in handling aspects of topology and number theory. Fürstenberger’s topological proof was straightforward due to the well-established PVS topology library, part of NASALib. Similarly, the proof using Fermat numbers benefited from the comprehensive number theory library in the PVS prelude. All that, conjugated with the typing system and the ability to define custom functions, made it possible to address the nuances of the infinitude of primes and formalize the proofs in a rigorous and structured manner. It also allowed addressing the omissions and imprecision in the original proofs and facilitated discovering simpler proof alternatives.

The proof using Fermat numbers (2.1) in “THE BOOK,” shows that any two Fermat numbers are relatively prime, basically proving by induction that

the recursive relation  $\prod_{k=0}^{n-1} F_k = F_n - 2$  holds. Consequently, the number 1 is

the only common divisor of two different Fermat numbers, since any Fermat number is odd. From that, it is concluded that there must be infinitely many primes, since there are infinitely many Fermat numbers. The last consequence is assumed without proof. Although it is intuitive, it is not completely trivial. Indeed, its proof has at least the same level of difficulty as the verification of the previous recursive relation. The mechanization allowed addressing various similar omissions, and provided a clear insight that the proofs of intuitive observations are not necessarily minor parts of the proving process. For the specific previously discussed consequence, for each Fermat number  $k$ , it was considered the minimum prime divisor of  $k$  (and formally proved that such a minimum exists). Then, an injective function was built from the set of natural numbers

to the set of minimum primes that divide Fermat numbers. Since this subset of prime numbers is infinite, the set of prime numbers itself is infinite.

Table 2 summarizes some relevant differences between the proof in “THE BOOK” and the mechanized proofs.

Further expansions of the presented formalization can include additional proofs uncovered in “Proofs from THE BOOK,” particularly those exploring other branches of mathematics or offering alternative perspectives on well-known approaches. One area of interest is the formalization of a geometry-related proof of the infinitude of primes, such as the one given in [10], which would broaden the scope of the library beyond number theory, analysis, topology, and algebra. Additionally, incorporating more advanced results in number theory, such as Dirichlet’s Theorem on primes in arithmetic progressions, would be a valuable addition. More in general, a key focus will also be to improve the level of automation in the PVS proofs. For instance, leveraging algebraic manipulations for structures other than number fields, which is currently highly automated through the Manip package [30]. This is particularly useful in streamlining the process of formalizing pen-and-paper proofs without obscuring essential mathematical reasoning steps.

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Table 2: Relevant differences between “Proofs from THE BOOK” on the infinitude of primes and their PVS formalization.

Proofs	“THE BOOK”	Formalization
<b>Fermat numbers</b> (Sec. 2.1)	Assumption of the infinitude of Fermat numbers implies the infinitude of primes, without a formal proof.	Build an injection from the infinite set of Fermat numbers to the set of minimal primes dividing each Fermat number.
<b>Mersenne Numbers</b> (Secs. 2.2 and 3.1)	Assumption of isomorphisms between algebraic structures.  Application of Lagrange’s Theorem.	Specification of particular types from cosets for applying results from NASALib regarding finite groups. Explicit proofs of properties and relations between modular arithmetic and quotient rings. Application of an algebraic lemma that considers the divisor of the order of an element, instead of using Lagrange’s Theorem.
<b>Euler product</b> (Secs. 2.3 and 3.2)	Informal usage of series of primes of the form $p_1, p_2 \dots$ , without explicitly considering the possibility of finitude.  Implicit usage of the Cauchy product formula.	Construction of an enumeration function of $\mathbb{P} \cup \{0\}$ avoiding the implicit assumption of the infinitude of primes. Alternative version of the Fundamental Theorem of Arithmetic, making explicit the power of each prime in the decomposition. Formalization of the Cauchy product.
<b>Fürstenberg</b> (Sec. 2.4)	Minor omissions when proving properties on the Cardinality of open and closed sets in the topological space built on sets $N_{a,b}$ .	Mechanized manipulation of different types of sets and formalizations of properties of the topological space using basic notions of the PVS set and topology theories.
<b>Prime reciprocal series</b> (Sec. 2.5)	Informal usage of notation $p_1, p_2, \dots$ for $\mathbb{P}$ .	Use of an explicit enumeration function. Further consideration on sets specified in terms of prime enumeration.

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